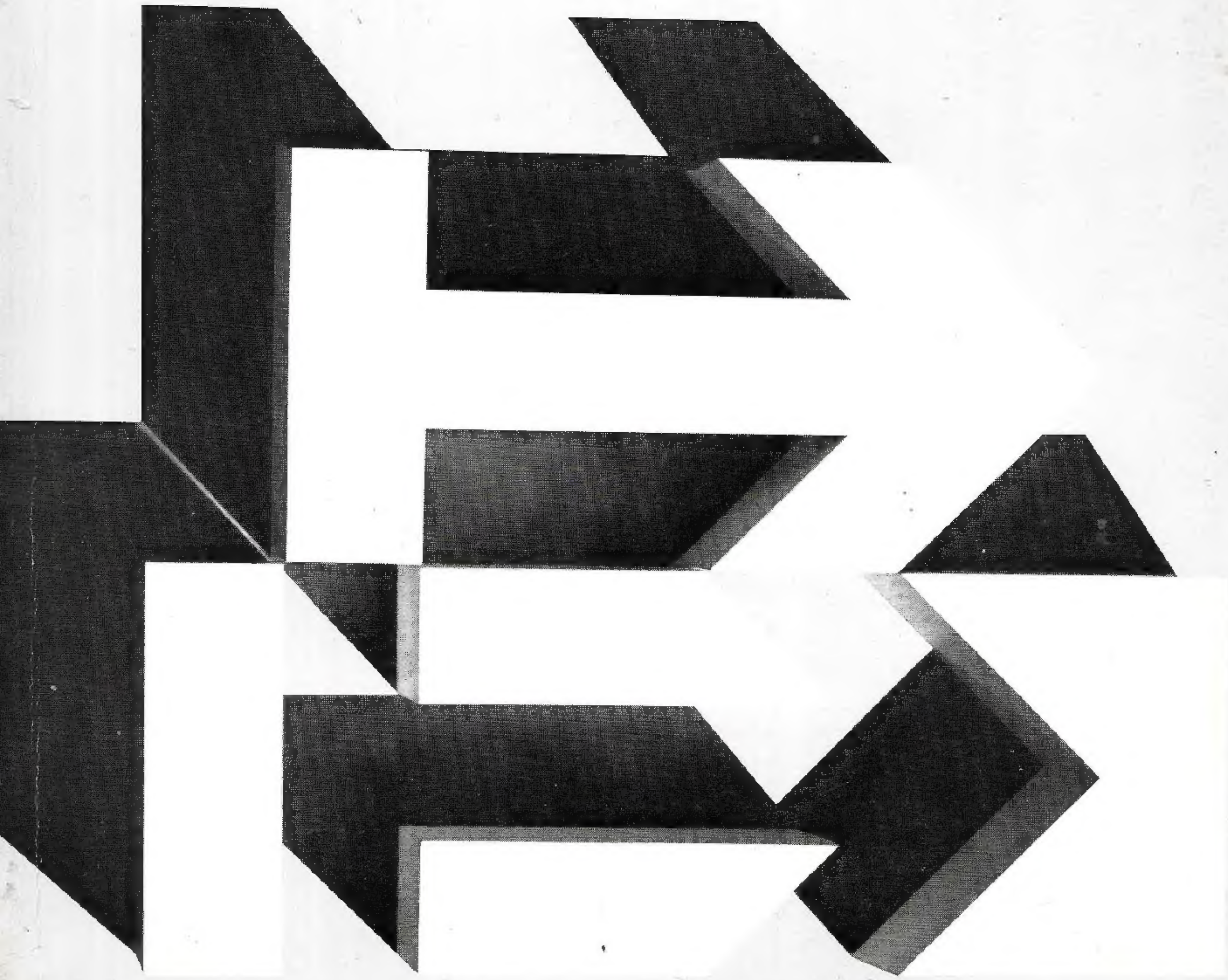
M100 1

THE OPEN UNIVERSITY

Mathematics Foundation Course Unit 1

Functions



The Open University

Mathematics Foundation Course Unit 1

FUNCTIONS

Prepared by the Mathematics Foundation Course Team

Correspondence Text 1

The Open University Press

The Open University Press Limited
Walton Hall, Bletchley, Bucks

First Published 1970

Printed in Great Britain by
J. W. Arrowsmith Ltd.. Bristol 3

SBN 335 01000 8

Open University courses provide a method of study for independent learners through an integrated teaching system, including textual material, radio and television programmes and short residential courses. This text is one of a series that make up the correspondence element of the Mathematics Foundation Course.

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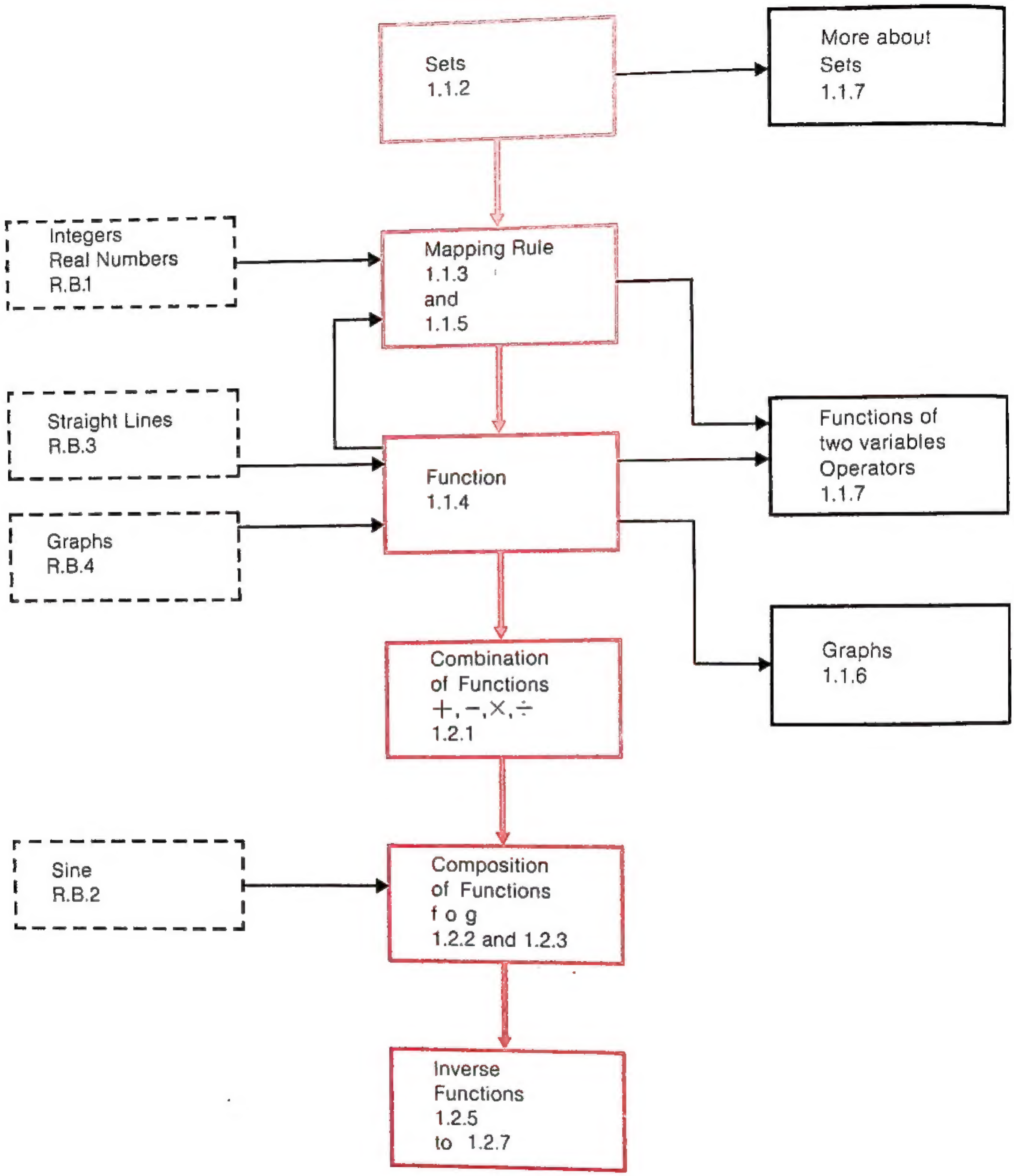
Contents

Objectives

After working through this unit you should be able to:

(i) give examples of mappings and functions with a clear statement of the domain, codomain and rule;
(ii) determine whether a given mapping is one-one, many-one, one-many or many-many;
(iii) distinguish between a function and a mapping;
(iv) calculate the images under a given mapping of elements in the domain and of subsets of the domain;
(v) combine given mappings by addition, subtraction, multiplication and division and by composition;
(vi) find the reverse of a given mapping.

Structural Diagram



Glossary

Terms which are defined in this glossary are printed in CAPITALS.

		Page
CODOMAIN	See MAPPING.	
CONSTANT FUNCTION	A CONSTANT FUNCTION is a FUNCTION for which every ELEMENT of the DOMAIN has the same IMAGE.	18
COMPOSITION OF FUNCTIONS	The COMPOSITION of two FUNCTIONS f and g is the combination of the functions to give the composite function $f \circ g : x \longmapsto f(g(x))$ $(x \in D)$, where D is the DOMAIN of g, when this is possible.	33
DIFFERENCE	If f and g are two FUNCTIONS with the same DOMAIN, A, and with codomain R, then their DIFFERENCE is the following function: $$f - g : x \longmapsto (f(x) - g(x))\ (x \in A).$$	30
DOMAIN	See MAPPING.	
DUMMY VARIABLE	A DUMMY VARIABLE is a VARIABLE used in the definition of a FUNCTION.	18
ELEMENT OR MEMBER	An object which belongs to a SET is called an ELEMENT or MEMBER of that set.	4
EQUALITY OF MAPPINGS	Two MAPPINGS are said to be EQUAL if they have the same DOMAIN and rule. (Strictly speaking, they should also have the same CODOMAIN, but we will not insist on this.)	17
EQUALITY OF SETS	Two SETS are said to be EQUAL if they have the same ELEMENTS.	4
FUNCTION	A FUNCTION is a MAPPING for which every ELEMENT of the DOMAIN has a single element as its IMAGE.	12
FUNCTION OF TWO VARIABLES	A FUNCTION of two VARIABLES is a function whose DOMAIN is a SET of ORDERED PAIRS (usually of numbers).	27
GRAPH	If f is a MAPPING, the SET of all pairs (x, y) such that x belongs to the DOMAIN of f and y is the IMAGE of x under f, or is an ELEMENT of that image, is the GRAPH of the mapping. When the pairs can be represented by points in a Cartesian plane the set of points is also called the graph of the mapping.	23
IMAGE	The IMAGE OF AN ELEMENT of the DOMAIN of a MAPPING is the element, or set of elements, assigned to it by the mapping. The IMAGE OF A SET, S, of elements of the domain is the set of all images of the elements of S.	7, 12
INVERSE FUNCTION	The INVERSE FUNCTION is the REVERSE MAPPING of a ONE–ONE FUNCTION.	44
MANY–MANY MAPPING	A MANY–MANY MAPPING is a mapping which is not a FUNCTION and whose REVERSE MAPPING is not a function.	41
MANY–ONE MAPPING	A MANY–ONE MAPPING is a FUNCTION whose REVERSE MAPPING is not a function.	41
MAPPING	A MAPPING consists of a SET A, called the DOMAIN, a set B, called the CODOMAIN, and a rule by which an ELEMENT or set of elements of B is assigned to each element of A.	11, 12

		Page
MEMBER	See ELEMENT.	25
MODULUS FUNCTION	The FUNCTION f: $$x \longmapsto \begin{cases} x \text{ if } x \geqslant 0 \\ -x \text{ if } x < 0 \end{cases} (x \in R)$$ is called the MODULUS FUNCTION.	
ONE–MANY MAPPING	A ONE–MANY MAPPING is a mapping which is not a FUNCTION, but whose REVERSE MAPPING is a function.	41
ONE–ONE MAPPING	A ONE–ONE MAPPING is a FUNCTION whose REVERSE MAPPING is a function.	41
OPERATOR	An OPERATOR is a MAPPING which has a SET of mappings as its DOMAIN.	29
ORDERED PAIR	An ORDERED PAIR is a pair of ELEMENTS, (a, b), which is, in general, regarded as distinct from the pair (b, a).	15
PROPER SUBSET	If a SET A is a SUBSET of B, but A is not EQUAL to B, then A is a PROPER SUBSET of B.	5
PRODUCT ~~QUOTIENT~~	If f and g are two FUNCTIONS with the same DOMAIN, A, and with codomain R, then their ~~QUOTIENT~~ PRODUCT is the following function: $$f \times g : x \longmapsto \frac{f(x) \times g(x)}{\cancel{g(x)}}, \quad (x \in A)$$ where the domain is the set $$\{x \in A : g(x) \neq 0\}.$$	30
REVERSE MAPPING	If a MAPPING f with DOMAIN A has GRAPH $$S = \{(x, y) : x \in A\}$$ and a mapping g has graph $\{(y, x) : (x, y) \in S\}$, then g is the REVERSE MAPPING of f and g has domain $f(A)$.	38
SET	A SET is any collection of objects.	4
SUBSET	If every ELEMENT of a SET A is also an element of a set B then A is a SUBSET of B.	5
SUM	If f and g are two FUNCTIONS with the same DOMAIN, A, and with codomain R, then their SUM is the following function: $$f + g : x \longmapsto \{f(x) + g(x)\} \ (x \in A)$$	30
VARIABLE	A VARIABLE is a letter used to stand for any ELEMENT of a SET. It is also called a DUMMY VARIABLE.	16

Notation

The symbols are presented in the order in which they appear in the text.

		Page
$\{a, b, c, d, \dots\}$	The set of elements $a, b, c, d, \dots$.	4
$A = B$	The sets A and B have the same elements.	4
$a \in A$	a is an element of the set A ("a belongs to A").	4
$A \subset B$	The set A is a proper subset of the set B.	5
$A \subseteq B$	The set A is a subset of the set B.	5
$f: A \longrightarrow B$	The mapping f maps the set A to the set B.	8
$f: x \longmapsto y$	The image of x under the mapping f is y.	8
$f(x)$	The image of x under the mapping f.	8
(a, b)	The ordered pair whose first member is a and second member is b.	14, 15
R	The set of real numbers.	15
$f: x \longmapsto y \ (x \in A)$	The mapping f has domain A and assigns to $x \in A$ the image y.	16
$f = g$	The mappings f and g have the same domain and the same rule.	17
R^+	The set of positive real numbers.	17
$[a, b]$	The set of real numbers x such that $a \leqslant x \leqslant b$.	24
$x \leqslant a$	x is less than or equal to a.	24
$x \geqslant a$	x is greater than or equal to a.	24
$\lvert x \rvert$	The modulus of x: $\lvert x \rvert = \begin{cases} x \text{ if } x \geqslant 0 \\ -x \text{ if } x < 0 \end{cases}$	25
$\{x : x \text{ has a property } P\}$	The set of *all* elements x which have the given property P.	27
$f \times g$	The product of two functions $f \times g: x \longmapsto f(x) \times g(x)$.	30
$f + g$	The sum of two functions $f + g: x \longmapsto f(x) + g(x)$.	30
$f - g$	The difference of two functions $f - g: x \longmapsto f(x) - g(x)$.	30
$f \div g$	The quotient of two functions $f \div g: x \longmapsto \dfrac{f(x)}{g(x)}$.	30
$g \circ f$	The function defined by $g \circ f: x \longmapsto g(f(x)) \ (x \in \text{domain of } f)$.	33

Bibliography

C. B. Allendoerfer and C. O. Oakley, *Principles of Mathematics*, 2nd ed. (McGraw-Hill, 1963).
Functions are discussed in Chapter 6. The terminology differs from ours in places but the slightly different approach may be helpful.

T. M. Apostol, *Calculus Vol. I*, (Blaisdell, 1961).
There is a section on functions commencing on page 50. The content is governed by the immediate needs of the remainder of the book and concentrates on functions with domains which are sets of real numbers.

J. G. Kemeny, H. Mirkil, J. L. Snell and G. L. Thompson, *Finite Mathematical Structures*, (Prentice-Hall, 1959).
If you feel well at home with the material in this unit, you may find some time spent with this book to be most rewarding. The approach is different from ours; so be careful.

1.0 INTRODUCTION

Before working through this correspondence text, make sure you have read the general introduction to the mathematics course in the Study Guide, as this explains the philosophy underlying the whole course. You should also be familiar with the section which explains how a text is constructed and the meanings attached to the stars and other symbols in the margin, as this will help you to find your way through the text.

The television programme for this unit explains some of the terms used: you should try to see this before getting down to detailed work on this text, particularly on Part 2.

The concept of "function", introduced in this unit, is fundamental to mathematics and is used in almost every subsequent unit of the course. In order to arrive at a clear idea of what is meant by this term, we have to spend a little time discussing the situations in which the idea of a function arises, and then a considerable amount of time defining the terms we need to describe the concept unambiguously. If you are not very familiar with mathematical thought, this may seem rather pernickety, particularly as many of the terms used are familiar in everyday language, and you may begin to wonder where we are going.

We wish to assure you, right from the start, that this rather long-winded approach is necessary. If we do not make quite sure what we are talking about from the very beginning, then sooner or later we shall get into difficulties or arrive at nonsensical conclusions. The history of mathematics is full of examples (some of which will be discussed in later units) of "mistakes" which have arisen in this way. In every case, the solution has been the same — go back to the beginning, re-examine the assumptions made and define the terms of the argument more carefully. This is particularly necessary when ideas have been uncritically taken over from everyday language. Time spent doing this at the beginning of your course is therefore time well spent: it will save you from errors later.

Even so, we cannot guarantee that if you pursue the subject far enough, you will not get into difficulties in the uncharted fields of new discovery and may eventually have to come back and re-examine these ideas once again, for this process of re-examination is an ever recurring one in mathematics and one which is still going on. This explains why some of the definitions and notations introduced in this unit may be rather different from those you have used in the past. Mathematics has progressed even in the last few years!

The second part of the text goes on to examine some of the ways in which functions may be combined to give more complicated functions, and to give you some practice in manipulating functions, all of which will be required in subsequent work.

Before reading the main text, have a look at the supplementary package. Besides a summary of the TV and radio programmes and the assessment questions, it may contain additional instructions which were prepared after the main text had gone to the printer.

Note

The solution to an exercise appears on the next non-facing page.

1.1 PART 1 MAPPINGS AND FUNCTIONS

1.1.0 Introduction

Introduction 1.1.0

In the first part of the text, our main concern is to explain and define the terms "mapping" and "function". In the first section the idea of mapping is discussed. Basic to the discussion is the notion of a "set", and when this idea comes to the fore, we devote a short section to introducing some notation which we shall be using throughout the course when we wish to refer to sets. We then go on to give formal definitions of "mapping" and "function". You should try to understand these definitions thoroughly, but it is not necessary to learn them by heart. As long as you *understand*, the details can be quickly checked when required by referring either to this section or to the glossary. You will find in time that continual use of the terms will make them familiar.

In the fifth section we discuss ways in which functions can be described and specified. Some functions can be illustrated by graphs, and so we conclude this part of the unit with a few remarks about graphs.

1.1.1 Groping for Definitions

Discussion 1.1.1

In this section we try to demonstrate the idea of mapping before making a precise definition. Although we make no definitions yet, we do introduce some words to describe the ideas under discussion.

Have a look at the following very simple examples, then we will try to extract the mathematical definition of "mapping" from them.

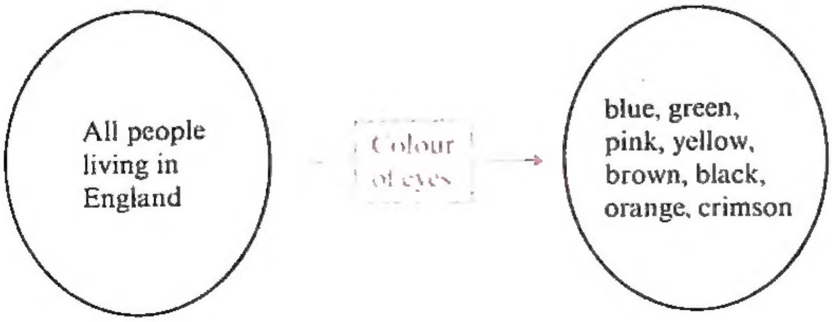
Example 1

Example 1

In this example a particular criminal corresponds (or we would say maps) to several of the fingerprints on record (since most people have more than one finger). Also all the fingerprints recorded in 1969 are only some of the fingerprints on record in 1971. ■

Example 2

Example 2



Discussion

If we take any person living in England we can use the rule Colour of eyes to associate one of the colours in the right hand box to him. For example:

Fred Smith → Colour of eyes → Blue

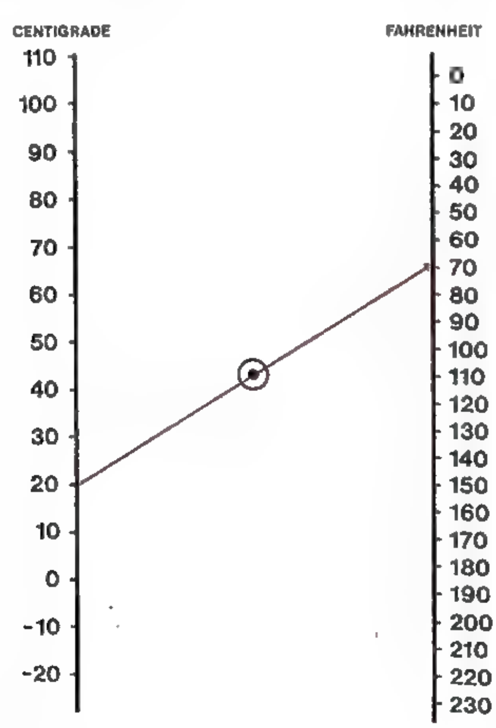
There are five things to notice about this example:

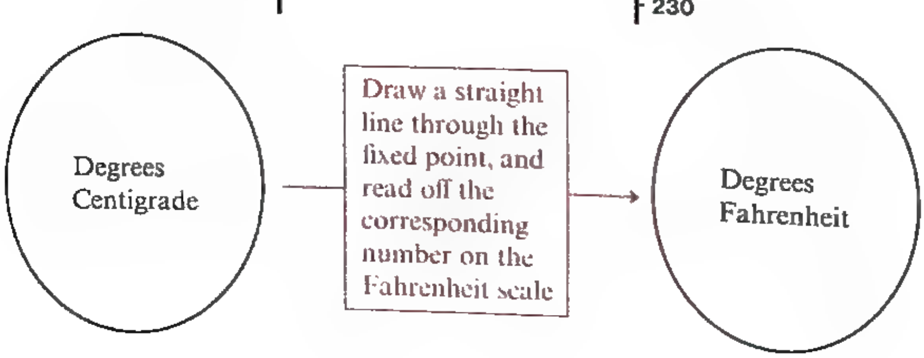
(i) We are told that the rule Colour of eyes applies to the set of all people living in England.
(ii) Each individual has a colour assigned to him by the rule (or possibly two colours, because there *are* rare individuals with eyes of different colours).
(iii) Many different people have the same colour assigned to them.
(iv) Not all of the colours on the right correspond to a person living in England. For example, crimson is listed even though no people have crimson eyes. You may have wondered why we included colours like crimson anyway. Strictly speaking we cannot be sure there are no crimson eyes *until* we have examined all the people. Perhaps there are other colours which *should* have been included?
(v) (There is a further point of a non-mathematical nature which ought to be mentioned. We are assuming that it is always easy to say exactly what colour the eyes are. In practice it may not be so simple. One person may say that your eyes are one colour, another person may say another. Some people even have eyes which appear to change colour in different lights.) ■

You may feel that the previous examples cannot really have anything to do with mathematics because we have not mentioned numbers. We are, of course, interested in numbers, but mathematicians are interested in a lot more besides.

Example 3

Example 3





Discussion

Once again we have a rule Draw the straight line etc. which we are told applies to the set of numbers on the Centigrade scale. For example:

$$0\,^{\circ}\text{Centigrade} \longrightarrow \boxed{\text{RULE}} \longrightarrow 32\,^{\circ}\text{Fahrenheit}$$

and from the diagram we have

$$20\,^{\circ}\text{Centigrade} \longrightarrow \boxed{\text{RULE}} \longrightarrow 68\,^{\circ}\text{Fahrenheit}$$

Try drawing some more straight lines on the diagram.

Notice particularly that in this example:

(i) Each temperature in °C corresponds to one, and only one, temperature in °F.

(ii) Each temperature in °F arises from one, and one only, temperature in °C. ■

The following situations are typical of many that arise in statistical surveys, scientific investigations and so on.

Example 4

Example 4

The *Highway Code* gives the following table for stopping distances of a car travelling at various speeds.

Speed (mile/h)	20	30	40	50	60
Stopping Distance (ft)	40	75	120	175	240

■

Example 5

Example 5

The following table gives the sales of electricity (in millions of kilowatt hours) for public lighting in Great Britain for the years 1938 to 1946.

Year	1938	1939	1940	1941	1942	1943	1944	1945	1946
Sale	367	248	17	18	20	20	28	177	260

■

Example 6

Example 6

The following table was obtained during an investigation of the variation in temperature along a wire carrying electricity. (The investigation was part of a research project aimed at developing a device for measuring electric current.) The distances are recorded in centimetres and measured from one end of the wire. The temperature is given in degrees Centigrade.

Distance	2	4	6	8	10
Temperature	25	42	50	51	44

(If you have no scientific background don't let the scientific details put you off. We are not interested primarily in the physical details, but in the mathematics which can be extracted from the physical situation. Examples such as these are included here and there because they illustrate the types of problems to which mathematics is relevant.) ■

1.1.2 Sets

These last three examples all have a feature in common. In each case pairs of numbers are recorded. One number of the pair is taken from one set of numbers and another from a second set of numbers. (In Example 1.1.1.4 they represent speed and stopping distance respectively.)

We have already had some examples which demonstrate the same idea and which do not involve numbers. But in each of the cases, one basic concept involved is that of correspondence between sets of objects. It is the idea of correspondence between elements of sets which is the main topic of this unit.

Definitions and Notation

Definition 1

One ordinary English word which we have used a few times already is the word set. We shall go on using the word in its ordinary, everyday sense to mean a collection of objects. So that we can refer more easily to particular sets we shall denote them by capital letters, such as A, B, ... For example, in Example 1.1.1.4 we have sets of numbers representing speeds and stopping distances. We could call them A and B, so that

A is the set $\{20, 30, 40, 50, 60\}$

and B is the set $\{40, 75, 120, 175, 240\}$.

Notation 1

Notice the way we write them: we list all the numbers of the set (*in any order*), separate them with commas, and enclose the whole lot with "curly brackets" (braces).

Definition 2

An object belonging to a set is called an element of the set (or sometimes a member of the set). We use a lower case letter to stand for an element, so we get statements like

"x is an element of X"

or just

"x belongs to X".

We use statements like this so often that we find it convenient to have a symbol in place of the words

"is an element of".

Notation 2

The symbol we use is $\in$, so that

"x is an element of X" becomes "$x \in X$"

but we still read it in the same way as before, or use the words "x belongs to X".

Two Important Points

Discussion

(i) *The order* in which the elements of a set are listed is *immaterial*, since we are interested in the set as a whole, and so we could equally well write

A as $\{20, 50, 30, 60, 40\}$

instead of $\{20, 30, 40, 50, 60\}$.

Definition 3

Two sets are said to be *equal* if they contain the same elements. Thus we write

$\{1, 2, 3\} = \{3, 1, 2\}$.

(ii) Each element of a set is listed *once only*. For example, the set of numbers in Example 1.1.1.5 which gives the sale of electricity is

$\{367, 248, 17, 18, 20, 28, 177, 260\}$.

The number 20 appears only once in the list, even though it occurs twice in the table.

In the same way in Example 1.1.1.2 we could list the colours:

PINK

{Blue, Green, ~~Red~~, Yellow, Brown, Black, Orange, Crimson}.

We list "Blue" once only, even though hundreds of people have blue eyes.

Subsets

Definitions and Notation

Any set of elements chosen from a set is called a **subset**. For example:

Definition 4

(i) $\{367, 20, 260\}$ is a subset of $\{367, 248, 17, 18, 20, 28, 177, 260\}$,
(ii) $\{a, b\}$ is a subset of $\{a, b, c\}$.

Strangely enough we say that

$\{a, b, c\}$ is a subset of $\{a, b, c\}$

but if we wish to say "a subset which is not just the original set" we say a **proper subset**.

Definition 5

To make the point clear:

$\{a, b\}$ is a proper subset of $\{a, b, c\}$;

$\{a, b, c\}$ is a subset, but not a proper subset, of $\{a, b, c\}$

We frequently use a symbol to stand for "is a subset of". We write

Notation 3

$A \subseteq B$

to stand for "the set A is a subset of the set B". For "is a proper subset of", we write

$A \subset B$

Notation 4

to indicate that A is a subset of B and that A is not equal to B.

Exercise 1

Exercise 1
(2 minutes)

In each of the following questions, indicate which (if any) statements are correct.

(i) If $A = \{367, 20, 260\}$ and $B = \{367, 248, 17, 18, 20, 28, 177, 260\}$, then

Space for your answer

(a) $A \subset B$
(b) $B \subseteq A$
(c) $A = B$
(d) $B \subset A$
(e) $A \subseteq B$
(f) A is a proper subset of B
(g) A is a subset of B

(ii) If $A = \{\text{Jim, Mary}\}$ and $B = \{\text{Blue, Green}\}$, then
(a) $A = B$
(b) $A \subset B$
(c) $B \subset A$

Solution 1

Solution 1

(i) (a), (e), (f), (g) are correct.
(ii) None of the statements is correct.

The fact that A and B have the same number of elements is not enough to give equality. For two sets to be equal, not only must they have the same number of elements but these elements must be the same. ■

1.1.3 Mappings

Discussion 1.1.3

Before our digression on sets we were talking about correspondences between elements of sets. We have called the sets in Example 1.1.1.4 A and B. We can describe the correspondence in various ways. We can say that

To each number in A a number in B is assigned

or

The set A is mapped to the set B

or, in symbols,

Notation 1

$$A \longrightarrow B$$

which we read as "A maps to B".

We can now make a first attempt at a definition of the term "mapping":

> The essential feature of a MAPPING is that we have two sets and a method of assigning to *each* element of one set one or more elements of the other.

Tentative Definition

Example 1

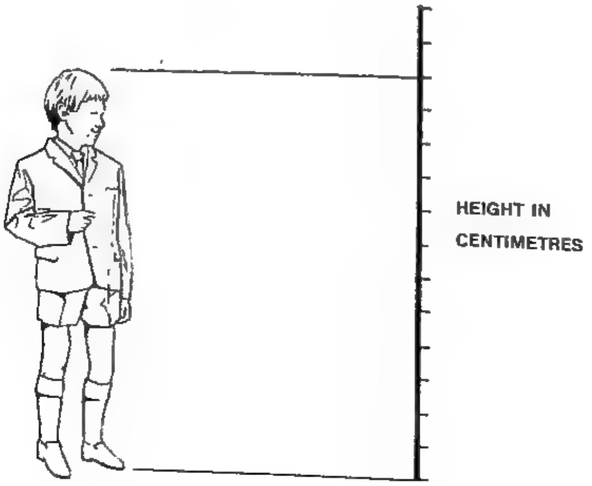
Example 1

We can map the set of all towns in the British Isles to the set of points on a piece of paper, by drawing a geographical map of the British Isles on the paper. ■

Example 2

Example 2

We can map the set of all people to the set of all integers, using the rule that to each person is assigned his height in centimetres measured to the nearest centimetre.



(Notice that there are numbers in the second set in this example which are not assigned to any person in the first set. We don't know of anybody 2 cm. high.) ■

Example 3

Example 3

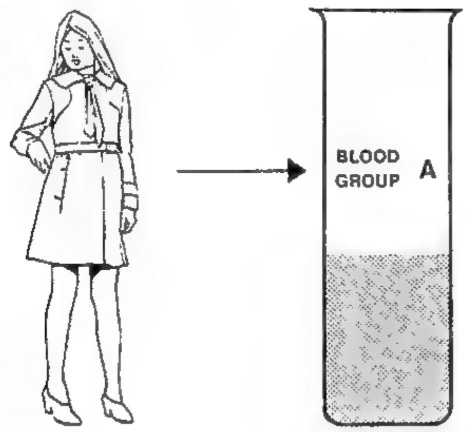
We can map the set of natural numbers to the set of natural numbers by assigning to each number its factors. For example:

6 maps to $\{1, 2, 3, 6\}$. ■

Example 4

Example 4

We can map the set of all people to the set of all blood-groups, by assigning to each person his blood-group.



■

It is convenient to have a shorthand notation for statements like "Elsie Tanner maps to Blood Group A."

Notation

We already have the notation

$$A \longrightarrow B$$

which stands for "set A maps to set B".

What we need is a notation which says that a *particular* element of A has assigned to it a *particular* element or set of elements of B.

If a belongs to A (i.e. $a \in A$) and b belongs to B (i.e. $b \in B$), the statement "b is assigned to a" is abbreviated to

$$a \longmapsto b$$

Notation 2

Instead of the precise "b is assigned to a" we also often say "a maps to b", where the context makes clear which we mean.

Notice that we have just added a small bar to the arrow to indicate that this is a precise assignment of elements rather than just a mapping of one set to another, where there may be elements in the second set which are not assigned to elements in the first. For example we should have had a barred arrow in the illustration of Example 4, but it would be incorrect to put a barred arrow in the statement $A \longrightarrow B$ unless we know that *every* element of B is assigned to an element of A. So in Example 2, $A \longmapsto B$ is incorrect, but in Example 4, $A \longmapsto B$ is correct.

Images

Definition 1

If $a \in A$, $b \in B$

and $a \longmapsto b$,

we say that b is the IMAGE of a.

For example, if

$$\text{Elsie Tanner} \longmapsto \text{Blood Group A}$$

then

Blood Group A is the image of Elsie Tanner.

This image of an element may be a set of elements, as we saw in Example 3, where a natural number has assigned to it its factors, e.g.

$$6 \longmapsto \{1, 2, 3, 6\}.$$

We could consider an image to be a set even if it consisted of only one element, but we do not make the distinction in this context between an element and the set ~~containing~~ COMPRISING that single element. For example, in the above we write $\{1, 2, 3, 6\}$ because when the image consists of more than one element it is convenient to think of it as a set. But although there is a logical distinction between Group A and {Group A}, for example, we shall ignore this distinction.

Naming a Mapping

If we wish to refer to a mapping, it is not always convenient to have to describe it in full every time, so we often use a symbol to represent it. For example, the mapping of the set of people to the set of blood groups could be represented by the letter h. We then write expressions like

$$h: \text{Set of people} \dashrightarrow \text{Set of blood groups}$$ Notation 3

involving whole sets, and

$$h: \text{Elsie Tanner} \longmapsto \text{Group A}$$ Notation 4

involving elements of sets.

Using the letter h we also write h (Elsie Tanner) to mean the image of Elsie Tanner under the mapping h. Therefore, we have:

$$h(\text{Elsie Tanner}) = \text{Group A}.$$ Notation 5

Either of the last two statements in red can be read "h maps Elsie Tanner to Group A" or "the image of Elsie Tanner under the mapping h is Group A", or "the mapping h, such that Elsie Tanner maps to Group A". (We shall use a colon to denote "such that" in other contexts later.) Any other convenient letter may, of course, be used in place of h provided that we define what it means.

Summary of Notation

Summary

$a \in A$	means	a is an element of set A.	Notation 1.1.2.2
$m: A \longrightarrow B$	means	The mapping m maps the set A to the set B.	Notation 1
$m: a \longmapsto b$	means	The mapping m maps the element a to the element b, OR	Notation 2, 4
		b is the image of a under (the mapping) m.	Definition 1

If the need arises we extend the notation still further and write, for example,

$$h: \{\text{set of all people}\} \longmapsto \{\text{set of all blood groups}\}$$

when every element of the image set corresponds to at least one element of the first set.

Discussion of Definitions and Notation

Discussion

The definitions and notation that we give are by no means the only ones possible. There is far more freedom in mathematics than is commonly thought. We choose our particular definitions because we consider them to be a workable set of definitions at our present level of discussion, and we have included only those which we need for our immediate purposes. Other people may use different definitions, notation and terminology, and you should be careful to check them in any book you read.

Choices like these arise as frequently in mathematics as in any other subject. In mathematics, however, we consider it important that once we have decided on a set of rules, we have got to play our game strictly in accordance with them; that is, until we change the rules, and declare new ones.

Example 5

Example 5

We reproduce the tables of Examples 1.1.1.4, 1.1.1.5 and 1.1.1.6 here for your convenience.

Table I

s

Speed (mile/h)	20	30	40	50	60
Distance (ft)	40	75	120	175	240

Table II

f

Year	1938	1939	1940	1941	1942	1943	1944	1945	1946
Sale	367	248	17	18	20	20	28	177	260

Table III

t

Distance	2	4	6	8	10
Temperature	25	42	50	51	44

The image of 20 under s is 40 and so we write

$$s: 20 \longmapsto 40$$

or

$$s(20) = 40$$

The image of 1943 under f is 20 and so we write

$$f: 1943 \longmapsto 20$$

or

$$f(1943) = 20$$

The image of $\{2, 4, 6\}$ under t is $\{25, 42, 50\}$ and so we write

$$t: \{2, 4, 6\} \longmapsto \{25, 42, 50\}$$

or

$$t(\{2, 4, 6\}) = \{25, 42, 50\}$$

If $A = \{2, 4, 6, 8, 10\}$ and $B = \{25, 42, 50, 51, 44\}$ we can write

$$t: A \longmapsto B \quad \text{and} \quad t(A) = B$$

But if $C = \{25, 42, 50, 51, 44, 99\}$, we *cannot* write $t: A \longmapsto C$ or $t(A) = C$ because 99 has no corresponding element in A, but we *can* write

$$t: A \longrightarrow C \quad \text{or} \quad t(A) \subseteq C$$

■

Exercise 1

Exercise 1
(3 minutes)

Complete the following, where s, f and t are the mappings in Example 5.

(i) The image of 30 under s is $\boxed{75}$

(ii) $t: 8 \longmapsto \boxed{51}$

(iii) $f: \boxed{1944} \longmapsto 28$

(iv) $f(1942) = \boxed{20}$

(v) $s(60) = \boxed{240}$

(vi) $t(\boxed{10}) = 44$

(vii) $s: \{30, 40, 50\} \longmapsto \boxed{75\ 120}$ 175

■

Exercise 2

Exercise 2
(2 minutes)

If m is a mapping which maps a set A to a set B, and if $a \in A$ and $b \in B$, which of the following statements are true and which false?

(i) If $m: a \longmapsto b$, then $m(a) = b$ TRUE/~~FALSE~~
(ii) If $m: a \longmapsto b$, then $m(a)$ is the image of b ~~TRUE~~/FALSE
(iii) If $m: a \longmapsto b$, then $m(a) \in B$ ~~TRUE/FALSE~~ TRUE

■

1.1.4 More Definitions

Definitions 1.1.4

There are several points still to tidy up if we wish to be more precise about the meaning of words such as MAPPING. In this section we introduce a few more technical words and give some formal definitions.

> If we have a mapping from a set A to a set B, then we call A the DOMAIN of the mapping, and B the CODOMAIN of the mapping.

> The mapping involves some method, or rule, whereby to **each** element of the domain an image is assigned, and the codomain contains all the images.

The Ingredients of a Mapping

Discussion

In Example 1.1.1.2 (the colour of eyes example) the domain is:

the set of all people living in England.

The codomain is:

{Blue, Green, Pink, Yellow, Brown, Black, Orange, Crimson}

and, for example,

the image of Fred Smith is Blue

In our first tentative definition of a mapping in 1.1.3 we required that to each element of the domain we must be able to assign an element (or elements) of the codomain. In other words each element of the domain must have an image in the codomain.

On the other hand, as we have seen in Example 1.1.1.2, there is no reason why the codomain should not include elements which are not images, for there is no question of applying a formula, or a rule, to elements of the codomain. Indeed, it is convenient not to insist that every element in the codomain be an image of an element in the domain. How, for example, could we predict the exact set of intelligence quotients (the codomain) of a set of people (the domain) before measuring them? All we know is that all the intelligence quotients will be found in the set of natural numbers, and so we could choose this set as the codomain. (See RB1)

All we require of the codomain is that it should contain the set of all images of elements in the domain. Thus, for instance, in the "colour of eyes" example we do no harm to include crimson as one of our colours, even though it is not the colour of anyone's eyes.

We are now in a position to make some definitions. *Notice that in the first definition we define a mapping to consist of three things (the domain, the rule, and a codomain).*

Solution 1.1.3.1

Solution 1.1.3.1

(i) 75
(ii) 51
(iii) 1944
(iv) 20
(v) 240
(vi) 10
(vii) {75, 120, 175} ■

Solution 1.1.3.2

Solution 1.1.3.2

(i) TRUE
(ii) FALSE
(iii) TRUE ($m(a)$ stands for the image of a under m. The image belongs to the set B, and so $m(a) \in B$.) ■

Summary of Definitions

Summary

Definition 1

A MAPPING consists of a set A, a set B and a rule by which an element (or set of elements) of B is assigned to *each* element of A.

Definition 2

The set A is the DOMAIN of the mapping.

Definition 3

The set B is the CODOMAIN of the mapping.

Definition 4 (Definition 1.1.3.1)

If an element a of the domain has assigned to it an element b or a set of elements T of the codomain, then b or T is the IMAGE of a. Each element of T is called *an* image of a. If T contains only one element, b, then b is *the* image of a.

Definition 5

If T is the set of all elements of the codomain which are images of elements of the subset S of the domain, then T is the IMAGE of S.

The mappings for which each element in the domain has a unique element as its image are particularly important, and so we give them a special name.

Definition 6

A FUNCTION is a mapping for which each element in the domain has only one element as its image.

Digression

Note

It is becoming increasingly common usage to use the terms "mapping" and "function" synonymously. What we have called a mapping is sometimes referred to as a "correspondence". We have kept to our definition of mapping because we feel that the word mapping carries with it more of a sense of movement from one set to another than the word correspondence. As with any mathematical term, you must be sure, if you consult a text book, of the way in which the author is using the term.

Exercise 1

Exercise 1
(5 minutes)

In each case state whether the statement is true or false:

(i) A mapping is always a function. ~~TRUE~~/FALSE

(ii) The domain of a mapping is the set of all images under the mapping. ~~TRUE~~/FALSE

(iii) If m is a function with domain A and codomain B, then $m(a)$ must be an element of B. $(a \in A)$ TRUE/~~FALSE~~

(iv) The set of all images under a mapping is called the codomain of the mapping. ~~TRUE/FALSE~~ FALSE

(v) If $A = \{\alpha, \beta, \gamma\}$ and $B = \{1, 2, 3\}$ then the list:

$$m:\alpha \longmapsto \{1, 2\}$$

$$m:\beta \longmapsto 1$$

$$m:\gamma \longmapsto 1$$

defines a mapping from A to B. TRUE/~~FALSE~~

(vi) The list in (v) defines a function. ~~TRUE~~/FALSE

(vii) If $A = \{\alpha, \beta, \gamma\}$ and $B = \{1, 2, 3\}$ then the list:

$$m:\alpha \longmapsto \{1, 2\}$$

$$m:\beta \longmapsto 3$$

defines a mapping from A to B. ~~TRUE~~/FALSE

■

Solution 1

Solution 1

(i) FALSE
A function is a special type of mapping. For example if A is the set of all television receivers in Great Britain and B is the set of all their valves, the rule which assigns to each television receiver its valves is a mapping, but not a function, because each receiver maps to numerous valves. But if we took as codomain the set of all television trade names, then the mapping of receivers to trade names is a function because each receiver has a unique trade name.

(ii) FALSE
This could be true in certain circumstances, if the domain and IMAGE SET ~~codomain~~ are the same sets, BUT see also part (iv), because there is something more to say about the codomain.

(iii) TRUE
By definition, the codomain contains all the images of the elements of the domain.

(iv) FALSE
The codomain may contain elements which are not images under the mapping.

(v) TRUE
The list satisfies the requirements of a "mapping rule". It assigns at least one element of B to *each* element of A.

(vi) FALSE
The mapping is not a function because α does not have a *unique* element as its image.

(vii) FALSE
No element is assigned to γ. ■

1.1.5 Ways of Specifying Mappings

Discussion 1.1.5

Having seen a definition of the term "mapping" we now want to see how we can apply this definition to construct particular mappings.

Mappings Defined by Lists

One point about our definition of mapping which you probably feel needs further amplification is the way in which we specify the rule which assigns images to the elements in the domain.

We have already seen one way: we can list all the elements of the domain, together with their images. Indeed, we have already seen this in Example 1.1.1.4. The set of the pairs of numbers

$$(20, 40) \quad (30, 75) \quad (40, 120) \quad (50, 175) \quad (60, 240)$$

each pair consisting of a speed followed by a stopping distance, could be understood to define a function.

Ordered Pairs

It is clearly important, when presented with a list like this, to know which numbers belong to the domain and which to the codomain. In the example, the first element of each pair comes from the domain and the second element is a corresponding element of the codomain.

We shall adopt this arrangement as a convention. When the order in which elements of a pair are written is important, the pair is usually called an ordered pair.

Definition 1

Rules for Constructing Mappings

Discussion

One very precise way of specifying a mapping is to write down a list of ordered pairs. This is fine, but it is obviously tedious or even impossible unless the list contains only a few elements. For example, the mapping of names to telephone numbers is usually given as a list. We call such a list a telephone directory; and we all know how large telephone directories can be.

We can sometimes overcome this difficulty by describing how the list could be constructed, rather than actually constructing it. Just think how useful it would be if we need only remember a simple rule which would give us someone's telephone number if we knew his name and address.

Any method of defining a mapping will be satisfactory if it enables us (theoretically at least) to construct the list, or to test whether or not a pair belongs to the list.

Example 1

Example 1

As an example of such a rule we could use:

> map a person to his height in centimetres measured to the nearest centimetre. ■

Rules for Mapping Numbers to Numbers

Discussion

The descriptive method of defining a mapping by a rule, rather than as a list, can be used when mapping numbers to numbers. For example we could use the rule

Double it

to define such a mapping. In the case of mappings of this kind the rule can often be abbreviated by using the common algebraic notation.

The $\boxed{\text{Double it}}$ mapping could be described by saying

For any element x in the domain $x \longmapsto 2x$

Example 2

Example 2

(See RB1)

Let f be the mapping with domain R (the set of all real numbers), and codomain R, defined by the rule:

If $x \in R$ then $f: x \longmapsto 6x^2 - 2x + 1$

(Throughout this course we use the symbol R to denote the set of all real numbers.)

We could equally well define this mapping (which is actually a function because the value of x defines the value of $6x^2 - 2x + 1$ *uniquely*) in words as follows:

> f is the mapping with domain the set of all real numbers, codomain the set of all real numbers, and the rule:
>
> for any real number (i) square it
> (ii) multiply the result by 6
> (iii) subtract from this result twice the original number
> and then (iv) add 1

■

We usually abbreviate the previous statements and, for example, refer to the mapping

$$f: x \longmapsto 6x^2 - 2x + 1 \quad (x \in R)$$

Notation 1

Any letter could have been used in place of x as an arbitrary element of the domain; however when the domain is R it is common practice to use x. Any letter used in this way, to make a statement about every element of a set, is called a VARIABLE.

Definition 2

The statement $x \in R$, in the formula above, contains rather a lot of information. It tells us that

(i) the domain is R

and

(ii) the letter x is a variable, which can take any value in the domain.

Example 3

Example 3

The statements

$$f: x \longmapsto 2x^2 - 3 \quad (x \in R)$$

$$f: t \longmapsto 2t^2 - 3 \quad (t \in R)$$

$$f: a \longmapsto 2a^2 - 3 \quad (a \in R)$$

are all equivalent, and all define the same function; each of the statements

$x \in R$

$t \in R$

$a \in R$

specifies the domain (as the set R), and each declares the corresponding letter to be a variable. On the other hand, the statements

$$f: x \longmapsto 2x^2 - 3$$

and

$$f: 6 \longmapsto 2 \times 6^2 - 3 = 69$$

do not define a mapping, for there is no mention of the domain. They are simply statements about what happens to *particular* elements under the mapping, in the first case the element x, in the second the element 6.

■

Whatever Happened to the Codomain?

Discussion

You will notice in our definition on page 12 that we must (among other things) specify the codomain in order to define a mapping, and yet we seem to be ignoring it. Is it true that:

$$f: x \longmapsto x^2 + 4x - 1 \quad (x \in R)$$

defines a mapping?

Strictly speaking the answer is NO, but if we assume that the codomain is also R then the definition is complete. Since, in general, given the domain and the rule we can work out the set of images (and the codomain can be any set containing the set of images), we shall not include the specification of the codomain in the definition of the mapping unless we have a particular interest in it.

Definition 3

Strictly speaking mappings are equal only if they have the same DOMAIN, CODOMAIN and RULE. However, we shall be content with having the same domain and rule. For example, in the mapping above

$$f: x \longmapsto x^2 + 4x - 1 \quad (x \in R)$$

we would not distinguish between the two cases where the codomain is R and where it is the set of real numbers greater than or equal to -5, which is the set of images. If f and g are two mappings with the same domain and rule, we shall write

Notation 2

$$f = g$$

Exercise 1
(5 minutes)

Exercise 1

For each of the functions f, g and h defined below, state

(a) the domain,
(b) the image of the set $\{1, 2, 3\}$,
(c) the image of the domain.

Notation 3

(i) $f: x \longmapsto 2x + 1$ ($x \in$ the set of positive real numbers, which is denoted by R^+),
(ii) $g: x \longmapsto x^2 - 2$ ($x \in R$),
(iii) $h: x \longmapsto 3$ ($x \in R$). ■

Exercise 2
(5 minutes)

Exercise 2

(If you have difficulty with this Exercise, it may help if you re-read Example 3.)

Say which of the following statements are true and which are false:

(i) The statement $f: x \longmapsto x^2 + 1$ ($x \in R$) implies that $f(2) = 5$ — TRUE/~~FALSE~~

(ii) The statement $f: x \longmapsto x^2 + 1$ implies that $f(1) = 2$ (Be careful!) — ~~TRUE/FALSE~~ FALSE

(iii) The statement $f(2) = 5$ implies that $f: x \longmapsto x^2 + 1$ ($x \in R$) — ~~TRUE~~/FALSE

(iv) The statement $f: x \longmapsto 2x + 6$ ($x \in R^+$) implies that $f(-10) = -14$ — ~~TRUE/FALSE~~ FALSE

(v) The statement $f: x \longmapsto 2x + 6$ ($x \in R$) implies that $f: t \longmapsto 2t + 6$ where t is any real number — TRUE/~~FALSE~~

(vi) The statements $f: x \longmapsto 6x - 1$ ($x \in R$) and $f: t \longmapsto 6t - 1$ ($t \in R$) are equivalent — TRUE/~~FALSE~~

(vii) The statements $f: x \longmapsto 4x^2 + 1$ ($x \in R$) and $f: t \longmapsto 4t^2 + 1$ ($t \in R^+$) are equivalent — ~~TRUE~~/FALSE ■

Solution 1

Solution 1

(i) (a) R^+ (b) $\{3, 5, 7\}$ (c) The set of real numbers greater than 1

(ii) (a) R (b) $\{-1, 2, 7\}$ (c) The set of real numbers greater than or equal to -2

(iii) (a) R (b) 3 (c) 3

This last function is an example of a constant function. A constant function is one for which the image of every element of the domain is the same.

Definition 4

What to Do if You Got it Wrong

If you got any of the part (a)'s wrong, have another look at page 16.

If you got any of the part (b)'s wrong and you do not think it was just an arithmetical slip, remember that in each case you have to substitute the numbers 1, 2 and 3 into the formula and list the results. The (c) parts are the hardest. Remember that you are not asked simply for a suitable codomain, but a precise specification of the set of all images. In (ii) (c), for example, you have to work out what range of values $x^2 - 2$ can take if x can be any real number. Since x^2 is always positive, or zero, $x^2 - 2$ can never be less than -2. Now have another try. ■

Solution 2

Solution 2

(i) TRUE

The statement defines f completely; 2 is in the domain and so we can substitute in the formula, and $2 \times 2 + 1 = 5$.

(ii) FALSE

The statement does not define f because the domain is not specified and so we do not know whether we are allowed to substitute the number 1 into the formula.

(iii) FALSE

f could be one of any number of functions which include 2 in the domain and which map 2 to 5, e.g.

$$f: x \longmapsto x^2 + 2x - 3 \quad (x \in R)$$

(iv) FALSE

-10 is not in the domain of f.

(v) TRUE

Since the domain is R, we *can* substitute any real number into the formula.

(vi) TRUE

This is an example to illustrate the fact that any letter can be used as a variable in the definition of a function. Sometimes such a letter is called a *dummy variable*.

Definition 5

(vii) FALSE

The domains are not the same. ■

1.1.6 Graphs

Discussion 1.1.6

In this section we relate the terms we have introduced so far to graphs, with which you may be familiar. We also introduce a notational device for specifying sets.

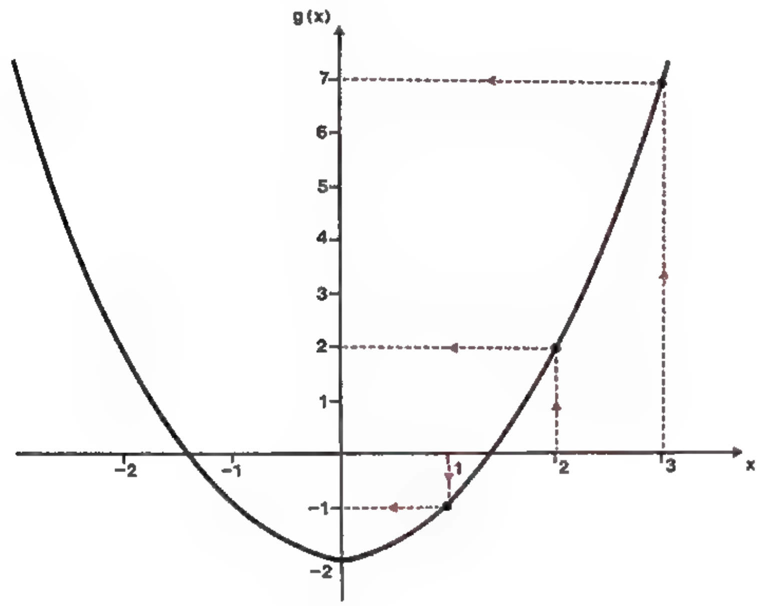
Example 1

Example 1

The function defined by

$$g : x \longmapsto x^2 - 2 \quad (x \in R)$$

has a graph which looks like this:



GRAPH of g

The mapping from the domain to the codomain can be visualized by following arrows drawn parallel to the vertical and horizontal axes:

$$g : 1 \longmapsto -1$$

$$g : 2 \longmapsto 2$$

$$g : 3 \longmapsto 7$$

Many mappings which are not functions can also be represented graphically. ■

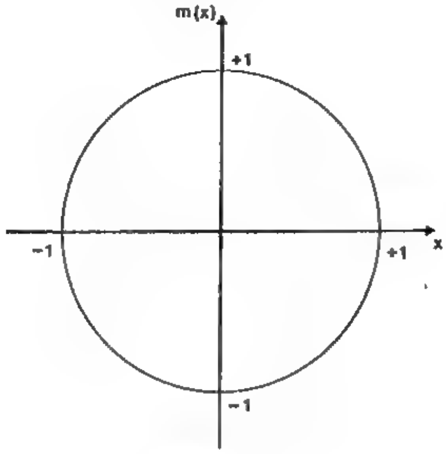
Example 2

Example 2

If $[-1, 1]$ is the set of real numbers between -1 and $+1$ inclusive, the mapping m defined by

$$m : x \longmapsto \{\sqrt{1 - x^2}, -\sqrt{1 - x^2}\} \quad (x \in [-1, 1])$$

is certainly not a function, because each element of the domain (except ± 1) has two images, but it has a graph which looks like this:



Graph of m

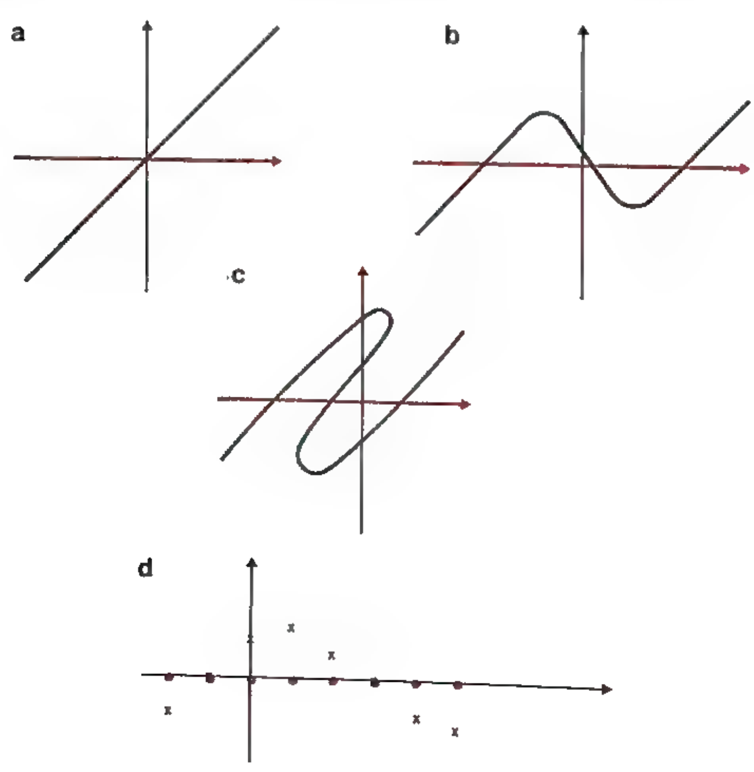
N.B.
In some older mathematical textbooks, the term function is used to cover expressions of the form $x \longmapsto \{\sqrt{1-x^2}, -\sqrt{1-x^2}\}$ $(x \in [-1, 1])$. This does not conform to the definitions adopted in this course. ■

Exercise 1

Exercise 1
(2 minutes)

Indicate which of the following graphs are
(i) graphs of functions
or
(ii) graphs of mappings that are not functions.

By convention, the horizontal axis is used to show the domain and the vertical axis is used to show the codomain. The domain is shown in red.



■

Cartesian Co-ordinates

Notation

As we have said before, it is conventional when drawing graphs of mappings, whose domains and codomains are sets of numbers, to represent the domain on a horizontal axis and the codomain on a vertical axis, and we shall use this convention. We have seen that it is common to use a letter x to stand for a variable in the domain and for this reason the horizontal axis is usually called the x-axis.

Similarly the letter y is often used as a variable in the codomain and so the vertical axis is called the y-axis.

(These axes are called Cartesian axes after their originator, the French mathematician and philosopher, René Descartes.)

René Descartes (Mansell)

Exercise 2

Exercise 2
(See Exercise 1.1.5.1)

Draw graphs for each of the following functions, and indicate the domain, the codomain and the image of the set $\{1, 2, 3\}$.

(See RB 3 and 4)

(i) $f: x \longmapsto 2x + 1 \quad (x \in R^+)$
(ii) $g: x \longmapsto x^2 - 2 \quad (x \in R)$
(iii) $h: x \longmapsto 3 \quad (x \in R)$ ∎

Solution 1

Solution 1

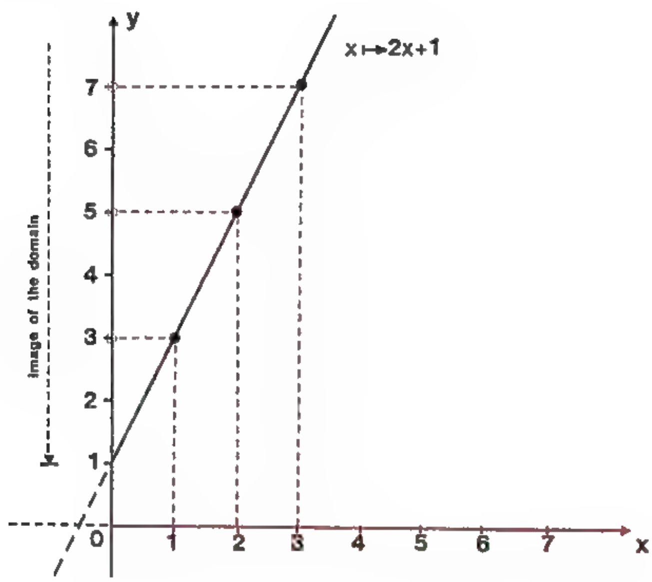
A function must specify only one image for each element of the domain, and so the graph of a function must do likewise — any line drawn parallel to the "codomain axis" must cut the curve at most once. Hence (c) does not specify a function.

(a) Function
(b) Function
(c) Not a function
(d) Function ■

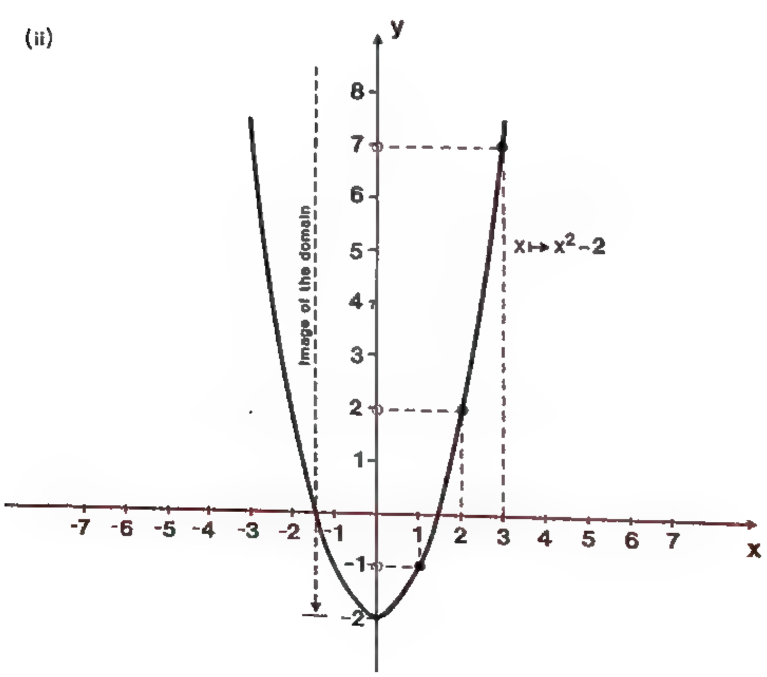
Solution 2

Solution 2

(i)

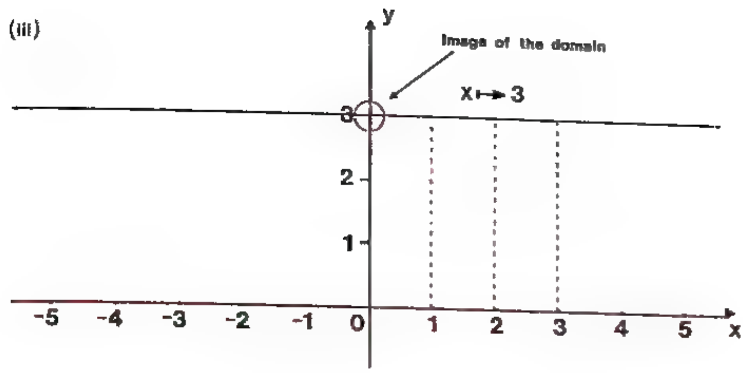


(ii)



Solution 2 (*continued*)

(iii)



■

What Do We Mean by a Graph?

Discussion

We have taken a GRAPH to mean a collection of points on a piece of paper; but unfortunately it is not always possible to represent mappings pictorially (in particular, if the domain and codomain are not sets of real numbers).

We shall be taking up this point again in *Unit 3, Operations and Morphisms*; we merely mention now, that to overcome these problems some people define the graph of the mapping f to be the set of all pairs of ~~numbers~~ ELEMENTS (x, y), where $x \in A$, the domain of f, and y is the image of x under f (or an image, where the image is a set).

Definition 1

We shall amplify these ideas when they are needed.

Using a Graph to Define a Function or Mapping

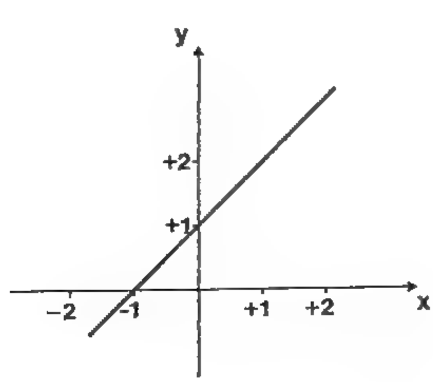
Discussion

Basically, we use a mapping to tell us how to draw a graph. Sometimes we can reverse the process and use a graph to help define a mapping.

Example 3

Example 3

The graph



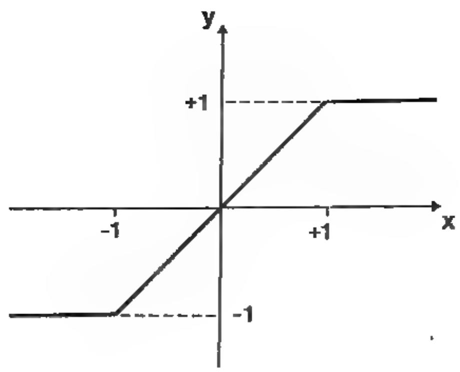
has the equation $y = x + 1$, and if we specify that x can take any real value in this equation then we have defined the function

$$f: x \longmapsto x + 1 \quad (x \in R)$$

■

Example 4

Example 4



This graph, together with a statement that x can take any real value, say, defines a function. But can we express this function by a formula? The formula is

$$y = -1 \quad \text{when } x \leqslant -1$$

$$y = +1 \quad \text{when } x \geqslant 1$$

and

$$y = x \qquad \text{when } x \in [-1, +1]$$

N.B.

(i) $x \leqslant a$ means "x is less than or equal to a" **Notation 1**

(ii) $x \geqslant b$ means "x is greater than or equal to b" **Notation 2**

(iii) $x \in [a, b]$ means "x belongs to the set of all real numbers between a and b, including a and b". **Notation 3**

These three symbols will be used frequently throughout the course.

Thus

$$\boxed{\begin{array}{ll} & f(x) = -1 \quad \text{if } x \leqslant -1 \\ & f(x) = +1 \quad \text{if } x \geqslant 1 \\ \text{and} & \\ & f(x) = x \qquad \text{if } x \in [-1, +1] \end{array}}$$

is the rule which defines the function f. The domain of f is R.
We could also write

$$\boxed{f: x \longmapsto \left\{\begin{array}{rl} -1 & \text{if } x \leqslant -1 \\ +1 & \text{if } x \geqslant +1 \\ x & \text{if } -1 \leqslant x \leqslant 1 \end{array}\right\} (x \in R)}$$

■

Exercise 3

Exercise 3
(2 minutes)

(i) Sketch the graph of the function f, where

$$f:x \longmapsto 1 + x \quad (x \in [-1, +1])$$

(ii) Sketch the graph of the function g, where

$$g:x \longmapsto 1 - x \quad (x \in [-1, +1])$$

You should have no difficulty in sketching these graphs very quickly free-hand on plain paper. ■

Exercise 4

Exercise 4
(2 minutes)

(i) Sketch the graph of the function

$$h:x \longmapsto \begin{Bmatrix} 1 + x & \text{if } x \geqslant 0 \\ 1 - x & \text{if } x \leqslant 0 \end{Bmatrix}$$

with domain $[-1, +1]$

(ii) Sketch the graph of the function

$$f:x \longmapsto \begin{Bmatrix} x & \text{if } x \geqslant 0 \\ -x & \text{if } x \leqslant 0 \end{Bmatrix} \quad (x \in R)$$

■

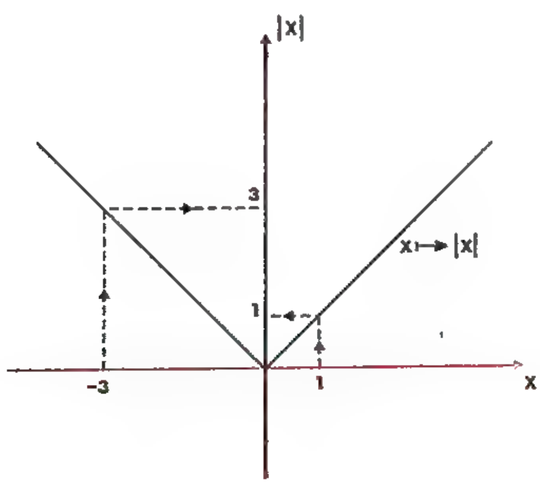
The MODULUS Function

The function in Exercise 4 (ii)

$$f:x \longmapsto \begin{Bmatrix} x & \text{if } x \geqslant 0 \\ -x & \text{if } x \leqslant 0 \end{Bmatrix} \quad (x \in R)$$

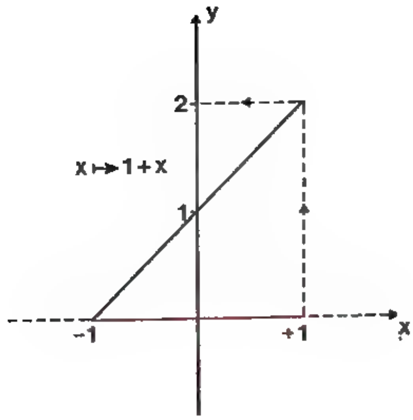
occurs very frequently in mathematics and so we give it a name and we have a special notation for it. We denote $f(x)$ by the symbol $|x|$ and call it the modulus of x. Thus, for example, $|1| = 1$ and $|-3| = 3$. The function is called the modulus function.

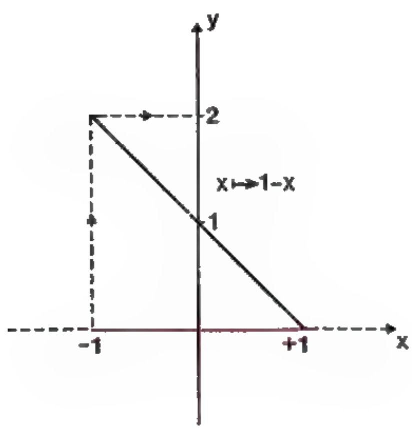
Notation 4
Definition 2



Solution 3

Solution 3





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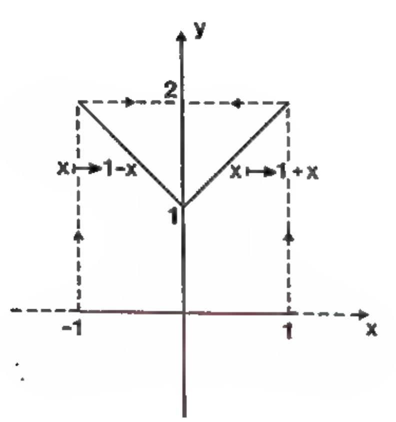
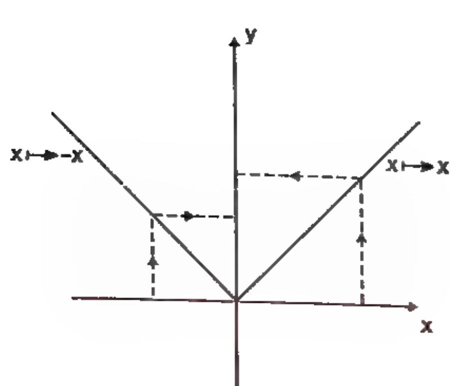




1.1.7 Further Notation and Definitions

Before we pass on to the second part of the text, we mention three short items which arise from our discussion so far. The first is an extension of our notation for sets, and the second and third items contain brief mentions of two types of function which we shall be meeting later in the course, and so include here for reference.

A Few More Words on Sets

We often want to refer to sets which have no conventionally accepted symbol to represent them. There is a standard way of writing such sets. A typical form of words is

"the set of all x such that x has some property"

This is conventionally written in mathematical shorthand as follows:

$\{x : x \text{ has some property}\}$ Notation 1

For example:

$\{x : x \text{ is a person with blue eyes}\}$

which we would read as:

The set of *all* x such that x is a person with blue eyes

Notice that the colon in the notation corresponds to "such that" in the reading.

For sets of numbers, we have, for example:

$\{x : x \in R \text{ and } x > 2\}$

which we read as:

The set of all x such that x belongs to the set of all real numbers and x is greater than 2

or, more briefly, as:

The set of all real numbers greater than 2

Functions of Two Variables

We have frequently emphasized that the domain of a function need not be a set of numbers. We have had domains which are sets of people, for example. There is one type of function which occurs quite frequently, and so we mention it briefly here. If the domain of a function is a set of ordered pairs, especially pairs of numbers, then we say that the function is A FUNCTION OF TWO VARIABLES. Definition 1

Example 1

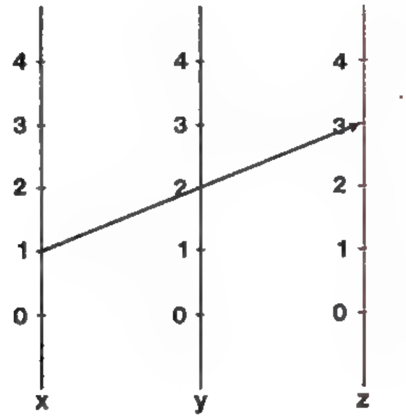
Example 1

$$f:(x, y) \longmapsto x^2 + y^2 \quad (x \in R \quad \text{and} \quad y \in R)$$

In this case f maps a pair of real numbers (x, y) to the sum of their squares. (You will see, in a later unit, how to illustrate functions like this.) ■

Example 2

Example 2



The pair (1, 2) maps to 3.
The diagram is in fact a device for calculating

$$2y - x$$

where x is represented by a point on the line **x**, and y by a point on the line **y**. The function under consideration is

$$f:(x, y) \longmapsto 2y - x \quad (x \in R \quad \text{and} \quad y \in R)$$

For example, if you draw the line through 3 on the **x** axis and 2 on the **y** axis, it will give 1 on the **z** axis, corresponding to $f:(3, 2) \longmapsto 1$. ■

Operators

We mentioned previously that we can refer to a mapping

$$m: A \longrightarrow B$$

simply by the letter m. There could be many different mappings from A to B, for example

$$m_1: A \longrightarrow B$$
$$m_2: A \longrightarrow B$$
$$m_3: A \longrightarrow B$$

In other words we could talk of the

$$\boxed{\text{set of mappings } \{m_1, m_2, m_3\}}$$

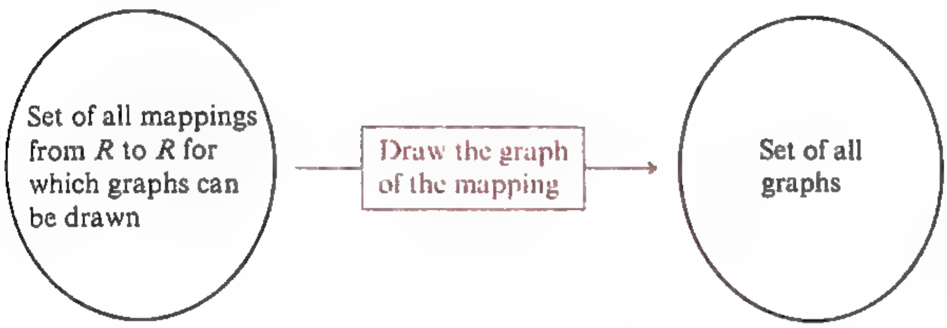
We have viewed the concept of the "domain" in a very flexible way so far. We have talked of mappings with all sorts of sets as their domain. Why not have a set of mappings as a domain? If there can be such things as sets of mappings, then we could conceive of *such sets being the domain of a mapping*.

This is quite a sophisticated idea. Just what have we said? We are used to talking about elements — numbers, colours, people, and so on. We are also quite familiar with dealing with collections, or sets, of elements. We are in the process in this unit of extending our ideas to think in some detail about correspondences between sets. Now we are considering such correspondences as elements of sets and talking about correspondences between correspondences. This is quite a conceptual jump — one which mathematicians have taken until this century to make, and one which has had a great influence on mathematics.

A mapping which has a set of mappings as its domain is sometimes called an OPERATOR, to distinguish its special nature. **Definition 2**

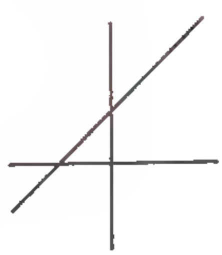
Example 3 **Example 3**

Let the domain be the set of all mappings from R to R for which graphs can be drawn and the codomain be the set of all graphs; and let the rule of the mapping be "draw the graph of the mapping".

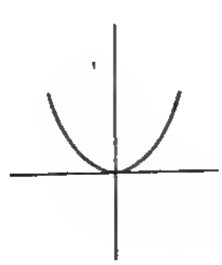


For one of the elements of the domain we would have

$(f: x \mapsto x+1\ (x \in R)) \mapsto$



$(g: x \mapsto x^2\ (x \in R)) \mapsto$



■

Summary of Part 1 **Summary**

We have introduced a large number of new terms. These are all listed in the glossary for easy reference until they become familiar. The really important idea is that of a mapping and, in particular, of a function. If you understand the terms, mapping, function, domain, codomain, set, and element, then you will be able to follow the second part of the text.

1.2 PART 2 COMBINATIONS OF FUNCTIONS

1.2.0 Introduction

Introduction 1.2.0

When one learns a new game, there are usually two stages to go through. First of all one has to find out something about the new materials — "take an ordinary pack of playing cards" or "take 22 men and a leather ball" or "take one egg, a cupful of flour and half a cupful of milk". The next step is to learn the rules of the game, that is to say, how to use the materials. We go through much the same process in mathematics. As soon as children grasp the idea of number, their teachers complicate the issue by showing them how to combine numbers by addition, multiplication, and so on.

This is the stage we are at now in this unit. We have defined our materials, and we are about to learn the rules of the game. More advanced tactics will be developed in subsequent units. The three items of particular importance in this part of the text are the arithmetical combination of functions, the composition of functions and the inversion of functions.

1.2.1 The "Arithmetic" of Functions

Definitions 1.2.1

Suppose two functions f and g have R as domain and codomain. Then we can define

Definition 1

$$\boxed{\text{the SUM of } f \text{ and } g}$$

which we write as $f + g$, by

$$\boxed{f + g : x \longmapsto f(x) + g(x) \quad (x \in R)}$$

Example 1

Example 1

Let

$$f : x \longmapsto x^2 \qquad (x \in R)$$

and

$$g : x \longmapsto x^6 \qquad (x \in R)$$

Then

$$f + g : x \longmapsto x^2 + x^6 \quad (x \in R)$$

■

It is quite natural to define the other "arithmetical" operations as follows:

DIFFERENCE

Definition 2

$$f - g : x \longmapsto f(x) - g(x) \quad (x \in R)$$

PRODUCT

Definition 3

$$f \times g : x \longmapsto f(x) \times g(x) \quad (x \in R)$$

QUOTIENT

Definition 4

$$f \div g : x \longmapsto \frac{f(x)}{g(x)}$$

The specification of the domain of the quotient is not straightforward. This is because of the difficulty which occurs when $g(x) = 0$. In this case the image of x is undefined, and we must remove such elements from the domain. So the domain of $f \div g$ is R with these elements omitted.

Exercise 1

Exercise 1
(2 minutes)

The first sentence of this section read: "Two functions f and g have R as domain and codomain". This is unnecessarily restrictive. Do both domain and codomain have to be R in order to define the operations of arithmetic used here? What must be true of the domains of f and g? Can you think of an example where either domain or codomain are not R? ■

Exercise 2

Exercise 2
(4 minutes)

If the functions f and g are defined by

$$f : x \longmapsto 6x^2 \quad (x \in [-1, 1])$$

and

$$g : x \longmapsto 6x \quad (x \in [-1, 1])$$

fill in the formula and the appropriate domain for

(i) $g + f$
(ii) $g \div f$
(iii) $f \div g$
(iv) $f \times g$ ■

Solution 1

Solution 1

If you look at the definitions, we do our arithmetic in the codomain. Therefore, only the codomain need be R. The only restriction on the domain is that it should be the same for both f and g. One example, among many possibilities would be

$f:x \longmapsto$ mark for first assignment $\quad$ ($x \in$ set of O.U.
$g:x \longmapsto$ mark for second assignment $\quad$ maths students) ■

Solution 2

Solution 2

(i) $g + f : x \longmapsto 6x + 6x^2 \quad (x \in [-1, 1])$
(ii) $g \div f : x \longmapsto 1/x \quad (x \in [-1, 1] \text{ and } x \neq 0)$
(iii) $f \div g : x \longmapsto x \quad (x \in [-1, 1] \text{ and } x \neq 0)$
(Although $f(x)/g(x) = x$, this is only true where we can actually perform the division. Therefore mathematically, and by definition, we must exclude $x = 0$. This is a substantial point, which is taken up in later units.)
(iv) $f \times g : x \longmapsto 36x^3 \quad (x \in [-1, 1])$ ■

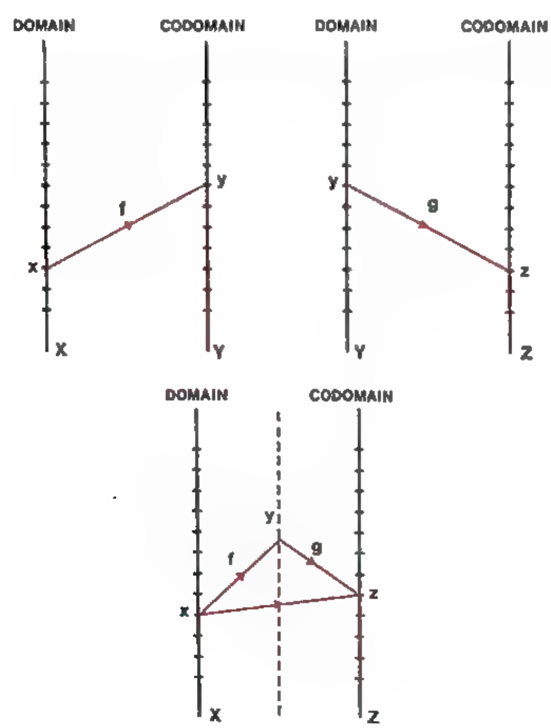
1.2.2 Composition of Functions

Discussion 1.2.2

There is another way of combining functions which is fundamentally different from the "arithmetical" combinations of the last section. The emphasis of this combination is on the mapping from one set to another, rather than being a simple generalization of the operations of ordinary arithmetic.

Example 1

Example 1



Notice that in the above example

$$f:\mathbf{x} \longrightarrow \mathbf{y} \quad \text{and} \quad g:\mathbf{y} \longrightarrow \mathbf{z}$$

The mapping in the bottom figure is obtained by using first f and then g. If we call it h, then

$$h:\mathbf{x} \longrightarrow \mathbf{z}$$ ■

Example 2

Example 2

Suppose that we have the functions

$$f = \boxed{\text{DOUBLE IT}}\ \text{with domain } R$$

$$g = \boxed{\text{SQUARE IT}}\ \text{with domain } R$$

then

$$x \longmapsto \boxed{\overset{f}{\text{DOUBLE IT}}} \longrightarrow 2x \quad (x \in R)$$

and

$$x \longmapsto \boxed{\overset{g}{\text{SQUARE IT}}} \longrightarrow x^2 \quad (x \in R)$$

Suppose now that we construct the function

$$x \longmapsto \boxed{\overset{f}{\text{DOUBLE IT}}} - 2x - \boxed{\overset{g}{\text{SQUARE IT}}} \longrightarrow 4x^2$$

by using *first* f and *then* g. If we call this *composite function* h, then

$$h: x \longmapsto 4x^2 \quad (x \in R)$$ ■

Composition

These examples each illustrate the same method of composition of functions. Unlike the operations of the last section, which were simply extensions of the operations of ordinary arithmetic, there is no analogue of this new composition in number arithmetic. We cannot extend the use of a symbol as we did for $+$, $-$, $\times$ and $\div$ because no such symbol exists; so we must invent one. The symbol which is commonly adopted is the small circle $\circ$. Definition 1

Thus $g \circ f$ (pronounced "gee oh eff") stands for the function obtained by performing f *first*, and *then* g. Notation 1

> If we can define a function h by the rule:
>
> $$h(x) = g(f(x)) \; (x \in \text{domain of } f)$$
>
> then we denote this function by
>
> $$h = g \circ f$$

(The only reason for introducing h into this definition is simply that we thought that

> $(g \circ f)(x) = g(f(x))$
> for all x in the domain of f

is not quite as clear. Sometimes ~~g(f(x))~~ $g \circ f$ is referred to as a function of a function.)

Diagrammatically we have

$$x \longmapsto f(x) \longmapsto g(f(x))$$

> It is very important to notice that $g \circ f$ means that we use f first and then g

Example 3 Example 3

If functions f and g are defined by

$$f: x \longmapsto 2x + 3 \quad (x \in R)$$

and

$$g: x \longmapsto x^2 - 1 \quad (x \in R)$$

we can calculate $g(f(x))$ by replacing x by $f(x)$ in the expression for $g(x)$:

$$g(x) = x^2 - 1$$

and so

$$g(f(x)) = [f(x)]^2 - 1$$

But

$$f(x) = 2x + 3$$

and so

$$g(f(x)) = (2x + 3)^2 - 1$$
$$= 4x^2 + 12x + 8$$

Thus $g \circ f$ is the function defined by

$$g \circ f: x \longmapsto 4x^2 + 12x + 8 \quad (x \in R)$$ ■

Exercise 1 Exercise 1 (2 minutes)

(i) If f and g are functions defined by

$$f: x \longmapsto x - 1 \quad (x \in R)$$

and

$$g: x \longmapsto x^2 \quad (x \in R)$$

fill in the boxes

(a) $f \circ g: x \longmapsto \boxed{} \quad (x \in R)$

and

(b) $g \circ f: x \longmapsto \boxed{} \quad (x \in R)$

(ii) If f is the mapping which translates English to French and g is the mapping which translates French to German is it $g \circ f$ or $f \circ g$ which translates English to German? ■

Exercise 2 Exercise 2 (4 minutes)

Given any two functions f and g,

(i) Can we always form $g \circ f$?

(ii) If we can form $g \circ f$, can we necessarily form $f \circ g$?

Explain your answers. ■

Exercise 3 Exercise 3 (2 minutes)

If both $f \circ g$ and $g \circ f$ are defined is it necessarily true that $f \circ g = g \circ f$? ■

Exercise 4

Exercise 4
(Omit if you got Exercise 1 correct)

If

$$f: x \longmapsto x^2 + 1 \quad (x \in R)$$

and

$$g: x \longmapsto 2x - 3 \quad (x \in R)$$

complete the following calculation of the function $g \circ f$ by filling in the boxes.

The image of R under f is the set of real numbers greater than or equal to 1, and this is contained in the domain of g. The combination $g \circ f$ is therefore possible.

We can specify $g \circ f$ by giving a formula for $g(f(x))$ and stating the domain.

(i) The domain of f is $\boxed{R}$ and therefore the domain of $g \circ f$ is $\boxed{R}$

(ii) Let $y = f(x)$, so that $y = \boxed{x^2+1}$

(iii) Let $z = g(y)$, so that $z = \boxed{2y-3}$

(iv) Substitute the formula of (ii) into the formula of (iii) to give $z = \boxed{2x^2-1}$

(v) $g \circ f$ is the function defined by $\boxed{2x^2-1}$ ■

Solution 1

Solution 1

(i) (a) $f \circ g : x \longmapsto x^2 - 1 \quad (x \in R)$
(b) $g \circ f : x \longmapsto (x - 1)^2 \quad (x \in R)$

(ii) It ought to be $g \circ f$, but in practice you would often get a different result going to German via French rather than directly from English. ■

Solution 2

Solution 2

(i) NO. We can form $g \circ f$ only if the set of all images of the domain of f is a subset of (or equal to) the domain of g. Consider, for example,

f: person $\longmapsto$ colour of his eyes (domain the set of all people)

$g : x \longmapsto x^2 \quad (x \in R)$

Then we cannot form $g \circ f$, since the square of a colour is not defined.

(ii) NO. We now have restrictions on the set of images under g and on the domain of f. Consider, for example,

$f : x \longmapsto \sqrt{x} \quad (x \in R^+)$

$g : x \longmapsto x + 3 \quad (x \in R)$

Then

$g \circ f : x \longmapsto \sqrt{x} + 3 \quad (x \in R^+)$ is satisfactory

but

$f \circ g : x \longmapsto \sqrt{(x + 3)}$ is undefined for $x < -3$. ■

Solution 3

Solution 3

NO. See, for example, Solution 1 (i). ■

Solution 4

Solution 4

(i) R, R
(ii) $y = x^2 + 1$
(iii) $z = 2y - 3$
(iv) $z = 2(x^2 + 1) - 3$
(v) $x \longmapsto 2x^2 - 1 \quad (x \in R)$ ■

1.2.3 Decomposition of Functions

Discussion 1.2.3

In the last section we were discussing methods of constructing formulas to describe composite functions. But it is often just as important to take a formula to pieces as to construct one. This is in fact just the process which is required when a formula is to be prepared for "digestion" by a computer.

Example 1

Example 1

Consider the simple function

$$f: x \longmapsto 2x + 1 \quad (x \in R)$$

and suppose that we are asked for $f(7)$. We reply 15 with hardly any thought at all. But how would we describe the calculation to a machine?

> INSTRUCTIONS
> (i) Multiply the number which I shall give you by 2.
> (ii) Add 1 to the result of (i).
> (iii) Print out the result of (ii).

We could, for example, describe this function f as a composition of two simpler functions g and h

$$h: x \longmapsto 2x \qquad (x \in R)$$

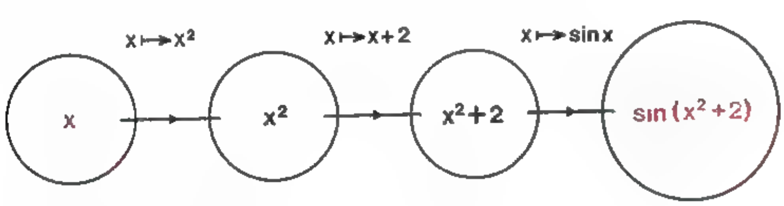
and

$$g: x \longmapsto x + 1 \qquad (x \in R)$$

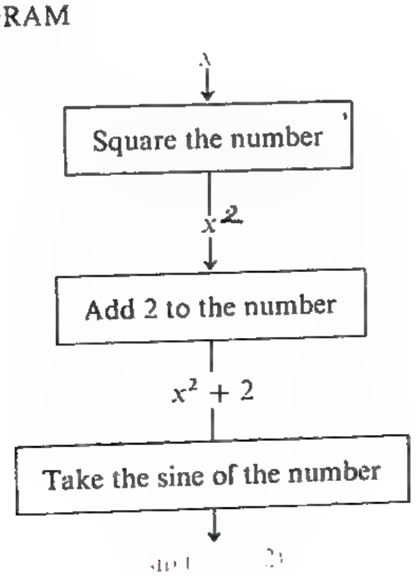
then

$$g \circ h: x \longmapsto 2x + 1 \quad (x \in R)$$ ■

One can imagine that for complicated functions this break-down of the calculation into simpler steps is an important part of numerical work. It is often worthwhile in complicated calculations to plan the steps either by using a suitable mapping diagram, for example



or a FLOW DIAGRAM



The ability to break down a complicated procedure into small units, whether it be a calculation or a logical organizational problem, is important in many fields. We shall meet the idea again, for example, in computing and in logic.

1.2.4 Reverse Mappings

Discussion 1.2.4

One of the examples which we used in the first chapter of this unit was the mapping of a set of people to their blood groups. This example illustrates a type of mapping, which often occurs, where it makes sense to reverse the mapping. In this case, the mapping is reversed when somebody needs a blood transfusion, and a donor has to be found. In this section we shall discuss this idea of reversing a mapping.

If one is trying to locate a book in a library, then the recommended procedure is to look it up in the catalogue and find its classification number: the classification system provides a mapping, c say, such that

$$c\colon \text{Books} \longrightarrow \text{Classification Numbers}$$

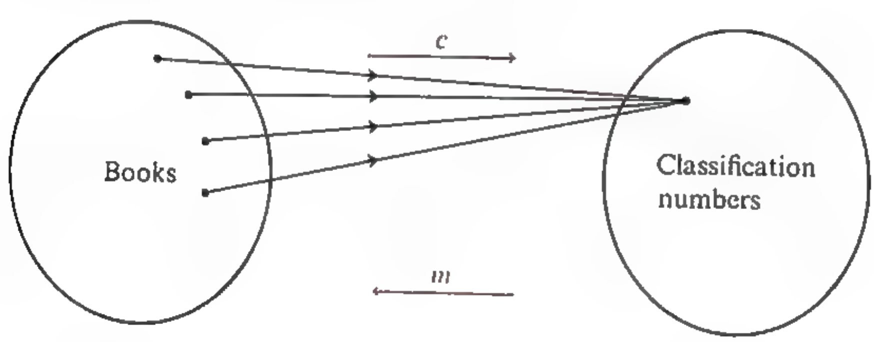
It is highly desirable that c is a function, rather than just a mapping. Can you say why?

On the other hand, if you want to find a book on a particular subject, and you know the classification number of the subject, then you look in a different catalogue which represents the mapping, m say, such that

$$m\colon \text{Classification Numbers} \longrightarrow \text{Books}$$

If m is a function rather than a mapping, then you can be sure that the library is not much good. Can you say why?

We say that m is the REVERSE MAPPING to c



A Definition

As we keep pointing out in this unit, it is not sufficient in mathematics just to understand a concept, we must be able to define it precisely. Probably the most convenient way to define a reverse mapping is in terms of the list of pairs which a mapping defines. A mapping f from A to B has a graph which is defined as the set of all pairs (x, y) such that $x \in A$ and y is $f(x)$ or, if $f(x)$ is a set of elements, belongs to $f(x)$.

Definition 1

> If f maps A to B and
>
> $$S = \{(x, y) : x \in A \quad \text{and} \quad y = f(x) \text{ or } y \in f(x)\}$$
>
> then the mapping of a subset of B to A, whose graph is
>
> $$\{(y, x) : (x, y) \in S\}$$
>
> is called the reverse mapping of f

Discussion

This definition is just a precise way of saying that we reverse the order of all pairs in the list which defines the mapping in order to get the reverse mapping. But one point is not entirely clear. What is the domain? We have said it is a subset of B. Consider the following examples.

Example 1

Example 1

Let us look again at the mapping of

Set of Persons to Set of Telephone Numbers

A telephone directory is a list of all pairs

(Person, Telephone Number)

If we reverse the order, to get a list of all pairs

(Telephone Number, Person)

This would represent the reverse mapping. ∎

Example 2

Example 2

Let us look once again at our "Colour of eyes" Example 1.1.1.2

Person living in England $\longmapsto$ Colour of his eyes

To reverse this we simply write

Colour of eyes $\longmapsto$ Person living in England with that colour of eyes

But we included CRIMSON in our list of colours and

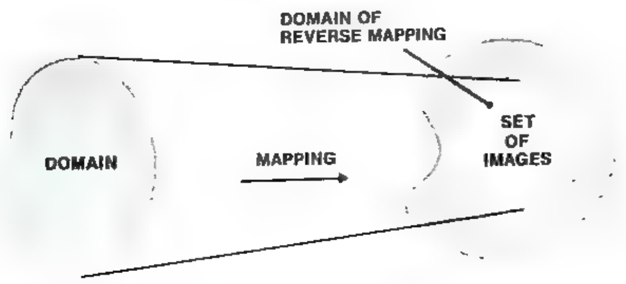
CRIMSON $\longmapsto$?

Is the reverse mapping in this case truly a mapping? What is the domain of the reverse mapping? (Look carefully at the definition of MAPPING if you are not sure how to answer these questions.)

You may well say that these difficulties were caused by our foolishness in including colours such as crimson in the codomain in the first place. But how are we to know what colours we ought to put in it before we examine the colour of everyone's eyes? ∎

Definition 2

> If g is the reverse mapping of f, and f has domain A, then the domain of g is $f(A)$



1.2.5 Classifying Mappings

Discussion 1.2.5

We have said previously that functions are important, but when is the reverse mapping a function? Certainly there is no guarantee that the reverse of a function is a function.

Example 1

Example 1

$$f: x \longmapsto x^2 \quad (x \in [-1, 1])$$

is a function.

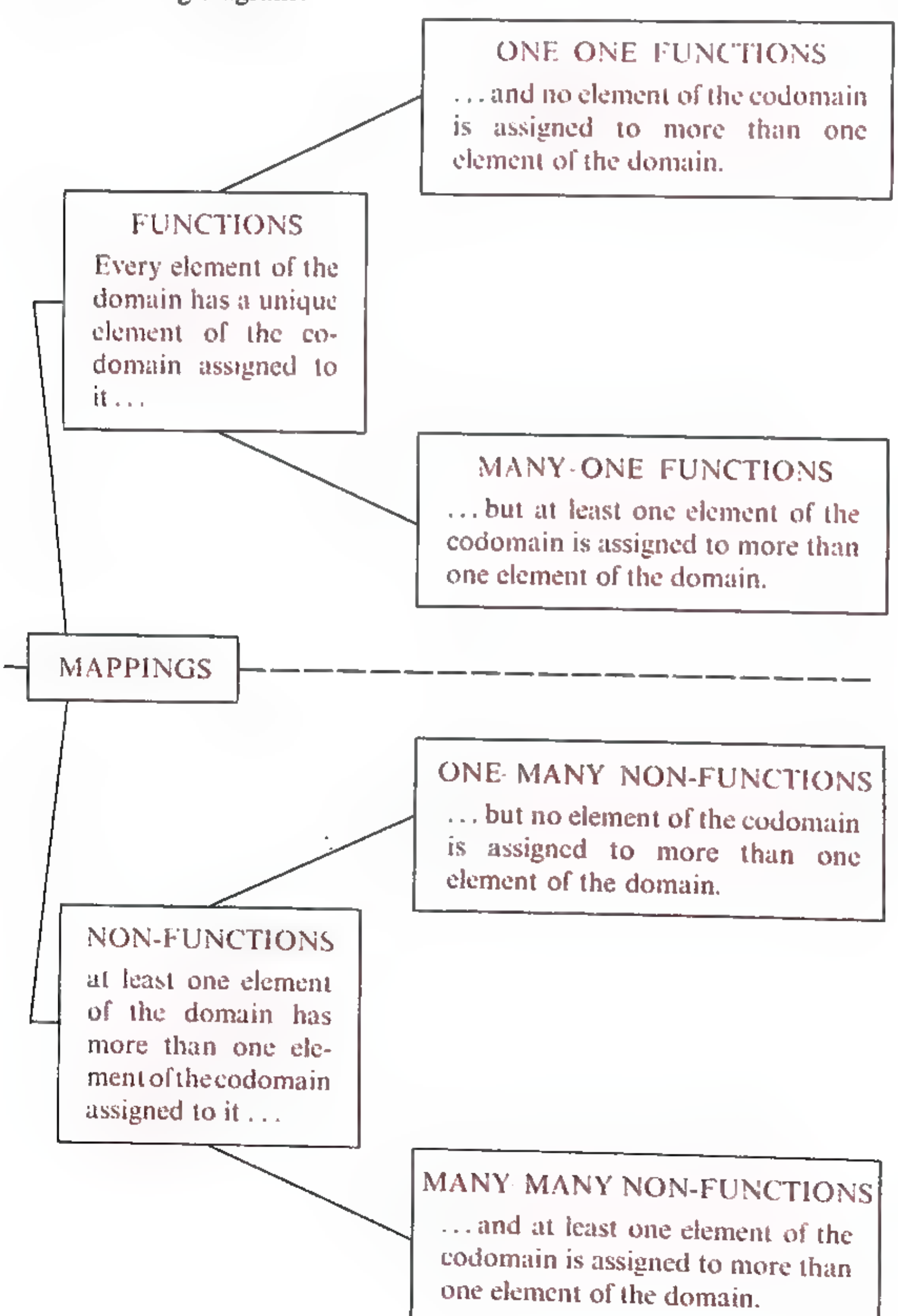
The reverse of f

$$g: x \longmapsto \{\sqrt{x}, -\sqrt{x}\} \quad (x \in [0, 1])$$

is not a function according to our definition, because each number in [0, 1] (except 0) has *two* separate square roots. ■

What sort of situations can arise? We have listed all the possibilities in the following diagram:

Definitions



Definition 1

Definition 2

Definition 3

Definition 4

This can all be put quite succinctly in terms of Definition 1.1.4.6 and Definition 1.2.4.1:

A one–one mapping is a function whose reverse mapping is also a function.

A many–one mapping is a function whose reverse mapping is not a function.

A one–many mapping is a mapping which is not a function, but whose reverse mapping is a function.

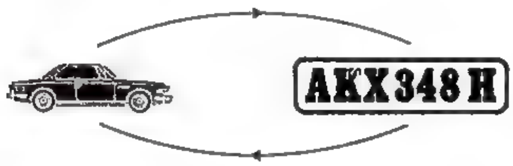
A many–many mapping is a mapping which is not a function, and whose reverse mapping is not a function.

Example 2

Example 2

ONE–ONE MAPPING (FUNCTION)

The above mapping is ONE–ONE (at any rate the authorities try to make sure that it is.)



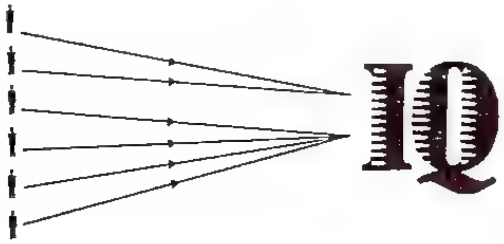
Each car has just one registration number. Each registration number corresponds to just one car. ■

Example 3

Example 3

MANY–ONE MAPPING (FUNCTION)

The mapping from the set of all people in Great Britain to the set of integers obtained by mapping a person to his intelligence quotient (on Jan. 1st 1971 for example) is MANY–ONE. Each person has a unique (i.e. just one) intelligence quotient (supposedly), but a large number of people map to 100, for example.

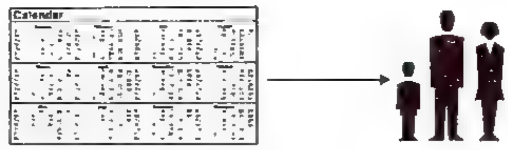


■

Example 4

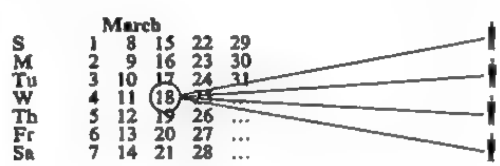
Example 4

ONE–MANY MAPPING (<u>NOT</u> A FUNCTION)



The mapping from the calendar to the set of all people living in Europe, obtained by mapping a date to the people born on that day, is ONE–MANY.

Each person has just one birthday, but many different people have the same birthday.



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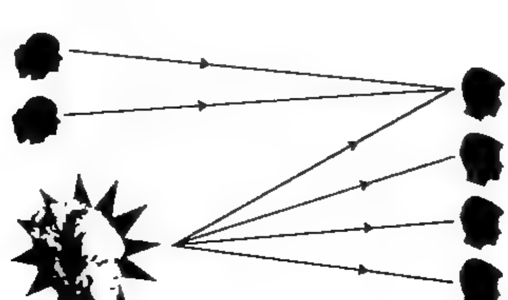
Example 5

Example 5

MANY–MANY MAPPING (<u>NOT</u> A FUNCTION)

The mapping from the set of all women who are or have been married to the set of all men, obtained by mapping a woman to her husband or ex-husband is MANY–MANY, because some people get divorced and re-married several times, and their partner could have been married before. (The fact that this is not true of everyone is irrelevant. If it is true for some people, that will be sufficient to make the mapping many–many.)



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Example 6

Example 6

The mapping defined by

$$f: x \longmapsto x^2 - 1 \qquad (x \in R)$$

is many–one (a function). $x^2 - 1$ is defined as just one number when x is given a value. On the other hand, an image may correspond to more than one number; for example, $f(3) = 8$ and $f(-3) = 8$.

$f: x \longmapsto x^2 - 1$ ■

Exercise 1

Exercise 1
(2 minutes)

Classify the following mappings as

$\boxed{\text{one–one}}$ or $\boxed{\text{many–one}}$ or $\boxed{\text{one–many}}$ or $\boxed{\text{many–many}}$

(i) $x \longmapsto 3x^2 + 2$ $\quad (x \in R)$
(ii) $x \longmapsto x^3 + 2$ $\quad (x \in R)$
(iii) $x \longmapsto \sin x$ $\quad (x \in R)$ (See RB2)
(iv) $x \longmapsto \{x, -x\}$ $\quad (x \in R^+)$
(v) $x \longmapsto \{\sqrt{(1 + 3x^2)}, -\sqrt{(1 + 3x^2)}\}$ $\quad (x \in R)$ ■

Solution 1

Solution 1

(i) Many–one, e.g. $1 \longmapsto 5$ and $-1 \longmapsto 5$.
(ii) One–one.
(iii) Many–one, e.g. $0 \longmapsto 0$, $\pi \longmapsto 0$, etc.
(iv) One–many. (Had the domain been R instead of R^+, the mapping would have been many–many.)
(v) Many–many, e.g. $1 \longmapsto \{2, -2\}$, $-1 \longmapsto \{2, -2\}$. ■

1.2.6 Inverse Functions

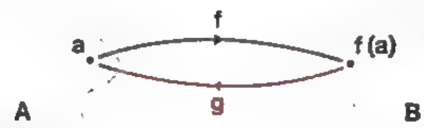
Discussion 1.2.6

We previously posed the question: "When is the reverse mapping a function?". We can now answer that question quite simply:

> The reverse mapping is a function if the original mapping is ONE–ONE or ONE–MANY. In both of these cases we can call the reverse mapping the reverse function.

The One–One Case

ONE–ONE functions are interesting, for if f maps A to B, and g is the reverse function of f: then f takes an element, a, in A to its image in B, and g brings this image back to a, (and to a only).



In other words

$$g(f(a)) = a \quad \text{for all } a \in A$$

Notice that this statement would also be true if f were a one–many mapping. But there is a difference; for one–one mappings we can *also* state

$$f(g(b)) = b \quad \text{for all } b \in f(A)$$

We now adopt the following definition:

> If f is a ONE–ONE function from A to B, and if $f(A) = B$ (i.e. the codomain of f is equal to the set of all images), then the function g from B to A where $g(f(a)) = a$ ($a \in A$) is called the INVERSE FUNCTION to f.

Definition 1

The condition which we impose on the codomain of f, namely that $f(A) = B$, is included to maintain the symmetry; so that f maps A to B, and g maps B to A. The condition $f(A) = B$ is equivalent to saying that B is the smallest set that will do the codomain's job; in other words, we do not want any odd elements like the unfortunate colour crimson in our "Colour of eyes" example.

For one–one functions whose domain and codomain are R, or subsets of R, we can often calculate inverses by algebraic manipulation.

Example 1

Example 1

Suppose that we wanted to calculate the inverse function of f where

$$f: x \longmapsto 3x + 2 \quad (x \in R)$$

If we put

$$y = f(x)$$

then

$$y = 3x + 2$$

and we can calculate y if we are given x.

The inverse function will enable us to calculate x if we are given y, and we can find this inverse g by rearranging the equation $y = 3x + 2$ to give an equation of the form

$$x = \text{something involving } y \text{ (and not } x) = g(y)$$

If we do this we get

$$x = \frac{y - 2}{3}$$

and so

$$g(y) = \frac{y - 2}{3}$$

and therefore g is the mapping

$$g: y \longmapsto \frac{y - 2}{3} \quad (y \in R)$$

We could of course rewrite this in the equivalent form

$$g: x \longmapsto \frac{x - 2}{3} \quad (x \in R)$$

■

Exercise 1

Exercise 1
(2 minutes)

Determine the inverse function of

$$f: x \longmapsto 4 - \frac{3}{x} \quad (x \in R^{+})$$

(Do not forget the domain of the inverse function.) ■

Exercise 2

Exercise 2
(3 minutes)

If g is the inverse function of the one–one function f, is it true that

(i) f is the inverse function of g? YES/NO
(ii) $g \circ f = f \circ g$? YES/NO
(iii) $g(x) = \dfrac{1}{f(x)}$? YES/NO

■

Exercise 3

Exercise 3
(3 minutes)

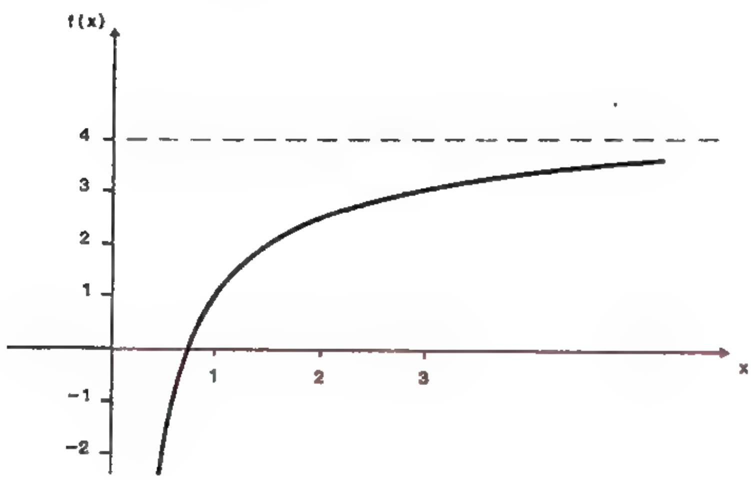
What can you say about the graph of a one–one function, assuming that it is an unbroken curve? (HINT: One approach to this might be to sketch a few graphs, decide whether or not they are one–one, and then try to generalize.) ■

Solution 1

Solution 1

$$g : x \longmapsto \frac{3}{4 - x} \quad (x \in \text{the set of real numbers less than } 4)$$

Remember that the domain is $f(R^+)$. Since x is positive, $4 - \frac{3}{x}$ is always less than 4. The graph of f shows quite clearly that the set of images is the set of all real numbers less than 4.



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Solution 2

Solution 2

(i) YES. We have actually defined the inverse function of f only when f is one–one. As we have hinted in the text, we could have defined an inverse function of a one–many mapping f; then of course, f is not a function.

(ii) Both $g \circ f$ and $f \circ g$ are given by the formula $x \longmapsto x$. But the domain of $g \circ f$ is that of f, and the domain of $f \circ g$ is that of g. So $f \circ g = g \circ f$ only if f and g have the same domain.

For instance, in the previous exercise you can check that the formula for both $g \circ f$ and $f \circ g$ is $x \longmapsto x$. But the domain of $g \circ f$ is the domain of f, i.e. R^+, whereas the domain of $f \circ g$ is the domain of g, i.e. the set of real numbers less than 4.

Example 1 provides a case where $g \circ f = f \circ g$.

(iii) NO. If $f : x \longmapsto x$, then $g : x \longmapsto x$, so $g(x) \neq \frac{1}{x}$ unless the domain of f is $\{1\}$.

■

Solution 3

Solution 3

The graph cannot "bend back" on itself. $f(x)$ must either increase ~~steadily~~ as x increases or decrease ~~steadily~~ as x increases.

■

1.2.7 Inversion of Composite Functions

Discussion 1.2.7

We have seen that the problem of finding an inverse function is essentially one of unravelling the original function. For functions from R to R this usually means unravelling a calculation. In some books this procedure goes under the name of "changing the subject of the formula".

It is sometimes useful, when finding inverses of relatively simple functions, to decompose a function into more elementary ones.

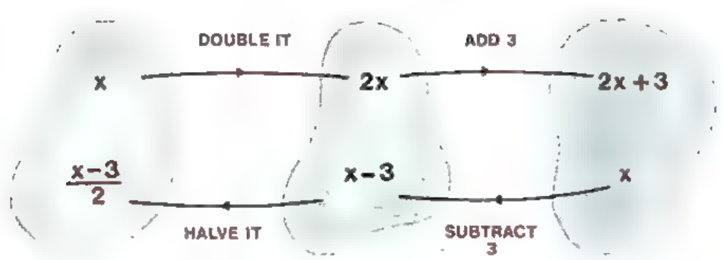
Example 1

Example 1

The function

$$f: x \longmapsto 2x + 3 \quad (x \in R)$$

has two components — "double it" and "add 3". If we want to invert this function we must unravel the calculation: "subtract 3" and "halve it".



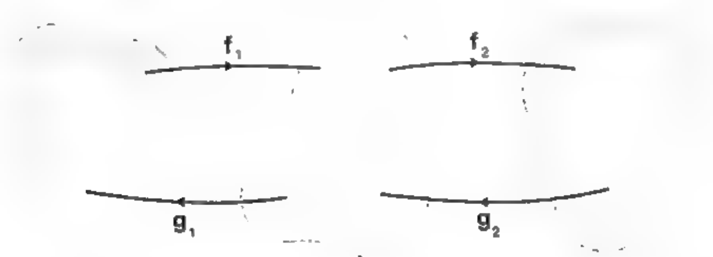
The inverse function is

$$g: x \longmapsto \frac{x-3}{2} \quad (x \in R)$$ ■

In general, if f_1 and f_2 are one–one functions and have inverses g_1 and g_2 then

Statement 1

$$\boxed{\text{the inverse of } f_2 \circ f_1 \text{ is } g_1 \circ g_2}$$



Note the order in which the inverses are combined — when inverting we have to invert the last step first. This is just like many operations in everyday life. When mending a puncture in a bicycle tyre, one removes the tyre first and then the inner tube. To invert this operation when the job is done, one replaces the inner tube first and then the tyre.

Exercise 1

Exercise 1
(2 minutes)

The one–one function f where

$$f: x \longmapsto 3x^2 + 2 \quad (x \in R^+)$$

maps an element x to an element y, where

$$y = 3x^2 + 2$$

The inverse function, g, will map to y back to x. By changing the subject of the formula, i.e. expressing x in terms of y, find a formula for g. ■

Exercise 2

Exercise 2
(2 minutes)

Calculate reverse mappings or inverse functions for the functions defined as follows:

(i) $f: x \longmapsto 7x - 1 \quad (x \in R)$
(ii) $f: x \longmapsto 4x^2 + 3 \quad (x \in R)$ ■

Solution 1

Solution 1

Rearranging the equation we get

$$x = \pm\sqrt{\left(\frac{y-2}{3}\right)}$$

Since f is one–one we must choose the signs correctly and get a single value for x in terms of y. Since $x \in R^+$, we choose the positive sign and so

$$x = \sqrt{\left(\frac{y-2}{3}\right)}$$

or

$$g : y \longmapsto \sqrt{\left(\frac{y-2}{3}\right)}$$

To find the domain of g, we require $f(R^+)$. If $x \in R^+$ then $f(x)$ is always greater than 2. Thus, we finally get

$$g : x \longmapsto \sqrt{\left(\frac{x-2}{3}\right)} \quad (x \in \text{the set of real numbers greater than } 2)$$

(Notice that this is an example where the domain of f is critical. If f had domain R, it would not be one–one, and the reverse mapping (*not* inverse) would not be a function.) ■

Solution 2

Solution 2

(i) $g : x \longmapsto \dfrac{x+1}{7} \qquad (x \in R)$

(ii) $g : x \longmapsto \left\{\sqrt{\left(\dfrac{x-3}{4}\right)}, -\sqrt{\left(\dfrac{x-3}{4}\right)}\right\} \qquad$ ($x \in$ the set of real numbers greater than or equal to 3)

■

SUMMARY AND CONCLUSIONS

Summary

We hope that you have enjoyed working through this first text, but now that you have got to the end it is possible that you may not appreciate the significance of all the ideas introduced. This is a perfectly natural reaction at this stage: after all, this is the first of 32 such texts, so don't be discouraged. When you have worked through four or five units you will begin to see the pattern and to appreciate the significance of some of these early ideas. Until we have developed the subject further, however, it is virtually impossible to demonstrate convincingly just how it makes sense.

We have introduced a large number of terms; you are not expected to remember most of these. The really important ones are

Set
Element
Domain
Codomain
Image
Mapping
Function
Graph
Composite function; $f \circ g$
Inverse function

Provided that you can handle these confidently, the other terms may be looked up in the glossary or notation sheets at the beginning of the text until they become familiar. The most important are also in the mathematical handbook. In most cases it is more important to be able to understand the terms when they are used in later work than to be able to reproduce their definitions.

You will find in the supplementary material accompanying this text a set of assessment questions for you to complete and return: these assessments form part of the continuous assessment which will be taken into account (together with the end-of-course examination) in the awarding of the course credit. They also serve the purpose of indicating to your tutor how well you are getting on so that he can give you any help which may be necessary. You may do the questions which refer to this unit at once, but please send them as instructed, and not in small bits.

The next unit does not continue the argument developed here although it makes use of some of the concepts. Instead, we switch to some practical considerations of importance in numerical evaluation. We shall come back to the fundamental mathematical ideas of this unit in *Unit 3, Operations and Morphisms.*

M100—MATHEMATICS FOUNDATION COURSE UNITS

1 Functions
2 Errors and Accuracy
3 Operations and Morphisms
4 Finite Differences
5 NO TEXT
6 Inequalities
7 Sequences and Limits I
8 Computing I
9 Integration I
10 NO TEXT
11 Logic I—Boolean Algebra
12 Differentiation I
13 Integration II
14 Sequences and Limits II
15 Differentiation II
16 Probability and Statistics I
17 Logic II—Proof
18 Probability and Statistics II
19 Relations
20 Computing II
21 Probability and Statistics III
22 Linear Algebra I
23 Linear Algebra II
24 Differential Equations I
25 NO TEXT
26 Linear Algebra III
27 Complex Numbers I
28 Linear Algebra IV
29 Complex Numbers II
30 Groups I
31 Differential Equations II
32 NO TEXT
33 Groups II
34 Number Systems
35 Topology
36 Mathematical Structures